# MATH-LITERACY MANUAL 

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## 3 Equations

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### 3.1 Equations vs. Expressions

An equation has two sides separated by equal sign. There is an expression on each side of the equation. As we have seen in Chapter 2, an expression never contains equal sign. This sign is used when transforming an expression to indicate that the newly obtained form is equal to the preceding one.

Example 1. What is given below, an equation or an expression?
(a) $7 x^{3}-4 y+1=5 x-6$
(b) $4 x^{2}-3 x-7=0$
(c) $4 x^{2}+2 x-7$
(d) $\frac{1}{z}=\frac{3 z}{z+5}$
(e) $\frac{1}{x}-\frac{3 x}{x+5}$
(f) $(x+1)^{2}=x^{2}+1$
(g) $t^{2}=-4$
(h) $t^{2}-4$

Solution. Equations are given under (a), (b), (d), (f), and (g). Expressions are in (c), (e), and (h).

Like expressions, equations may have more than one variable. This is the case with (a) above, which is an equation in two variables, $x$ and $y$. To the end of this chapter, we shall only consider equations in one variable.

Solving an equation and transforming an expression are different types of problems. When an expression is transformed, the initial and final expressions are equal for all real values of the variable (for which both expressions are defined). Equations, on the other hand, are conditional equalities - their sides do not have to be equal for all real numbers, but only for some specific values (if any) of the variable. A frequent mathematical problem is to find those values, that is, to solve the equation. In this context, the variable is often called the unknown. All solutions should be found. They form the solution set of the equation.

When an expression is to be transformed, the answer is just another form of the original expression. When an equation is to be solved, the answer is the solution set. For instance, the solution set for equation (b) in Example 1 is $\left\{\frac{7}{4},-1\right\}$. This means that the solution is $x \in\left\{-1, \frac{7}{4}\right\}$, which can also be written simply as $x=-1, x=\frac{7}{4}$, or as $x=-1, \frac{7}{4}$. When there is no solution, the answer is "no solution", or we may write the empty set $\emptyset$ for the solution set. When all real numbers solve the equation, we can write "all reals" or use $\mathbb{R}$ for the solution set.

Example 2. Which of the following is an acceptable answer when solving an equation for a real variable $x$ ? Sets given below represent solution sets.
(a) $x=\{-5,5\}$
(b) $x=0,-1,8$
(c) $x+4=0$
(d) $\left\{\frac{3}{7}\right\}$
(e) all non-negative real numbers
(f) $x=\pi$
(g) $x(x-1)$
(h) $\emptyset$

Solution. Only (a), (c) and (g) are unacceptable answers. (c) is not solved for $x$ and (g) is an expression. The form used in (a) is not correct because there is a real number $x$ on the left side and a set on the right side of the equality. These are two different mathematical objects which cannot be equal. The correct form would be $x \in\{-5,5\}$, or simply one of the following: $\{-5,5\}$; $x=-5, x=5 ; x=-5,5 ; x= \pm 5$. Answers (e) and (f) may look strange, but are possible. For instance, (e) is the answer when solving $\sqrt{x^{2}}=x$ and (f) solves $(x-\pi)^{3}=0$.

Expressions are equal, equations are equivalent. In a simplification problem (see Section 2.3), the given expression is transformed step by step and equal sign is used between consecutive steps. Thus,

$$
\begin{align*}
& 2(3 x-1)-4(x-3)  \tag{1}\\
& =6 x-2-4 x+12 \\
& =2 x+10
\end{align*}
$$

It is a typical mistake to apply the same notation to equations. For instance, if the equation $2(3 x-1)-4(x-3)=0$ is to be solved for $x$, the following is sometimes written by analogy to (1):

$$
\begin{align*}
& 2(3 x-1)-4(x-3)=0  \tag{2}\\
= & 6 x-2-4 x+12=0 \\
= & 2 x+10=0 \\
= & 2 x=-10 \\
= & x=-5 \text { WRONG! }
\end{align*}
$$

Why is this wrong? When we have a chain of equalities, this means that any expression in the chain is equal to any other expression in the same chain. In (2), $0,-10$, and -5 are parts of the chain, among other expressions. This means that $0,-10$, and -5 should be equal numbers, but, obviously they are not. The equal signs before each equation should be omitted in the correct
notation:

$$
\begin{align*}
2(3 x-1)-4(x-3) & =0  \tag{3}\\
6 x-2-4 x+12 & =0 \\
2 x+10 & =0 \\
2 x & =-10 \\
x & =-5
\end{align*}
$$

During the solution process, an equation is transformed step by step. Each step results in a new equation which is not equal but equivalent to the previous one. Equivalent equations have same solutions. Thus, equation $2 x+10=0$ in step 2 of solving (3) is transformed to $2 x=-10$ in step 3 . There is no way these two equations can be "equal" in some sense. All we are saying is that if $2 x+10=0$, then and only then $2 x=-10$. If we subtract 10 from both sides of $2 x+10=0$, we get $2 x=-10$, and if we add 10 to both sides of $2 x=-10$, we get $2 x+10=0$. Therefore, these two equations are equivalent. If we want to write something before each newly transformed equation, this should not be equal sign but the symbol for equivalent, $\Leftrightarrow$. Thus, (3) means the same as

$$
\begin{aligned}
& 2(3 x-1)-4(x-3)=0 \\
& \Leftrightarrow \quad 6 x-2-4 x+12=0 \\
& \Leftrightarrow 2 x+10=0 \\
& \Leftrightarrow 2 x=-10 \\
& \Leftrightarrow x=-5
\end{aligned}
$$

To illustrate this further, consider the equation

$$
\begin{equation*}
\frac{x}{4}+\frac{1}{3}=0 . \tag{4}
\end{equation*}
$$

One way of solving this equation is to clear fractions from it. Therefore, we multiply both sides by $\mathrm{LCD}=12$ to get

$$
12\left(\frac{x}{4}+\frac{1}{3}\right)=0
$$

When 12 is distributed over the two terms in parentheses, the fractions can be cleared:

$$
\begin{equation*}
3 x+4=0 . \tag{5}
\end{equation*}
$$

Now, in what sense can this be "equal" to (4)? Expressions $\frac{x}{4}+\frac{1}{3}$ and $3 x+4$ are not equal (recall from Section 2.3 that fractions cannot be removed from expressions since this changes the numerical value of the expression). The only thing we can say is that equations (4) and (5) are equivalent: if (4) is true, so is (5), and the other way around.

Example 3. For each problem given below, solution steps are provided, but some equal signs, numbers, or parts of expressions may be omitted. Insert what is missing.
(a) Factor $2 x^{3}+2 x^{2}-12 x$.

$$
\begin{gathered}
2 x^{3}+2 x^{2}-12 x \\
2 x\left(x^{2}+x-6\right) \\
2 x(x-2)(x+3)
\end{gathered}
$$

(b) Solve $2 x^{3}+2 x^{2}-12 x=0$ by factoring.

$$
\begin{gathered}
2 x\left(x^{2}+x-6\right) \\
2 x(x-2)(x+3) \\
2 x=0, \quad x-2=0, \quad x+3=0 \\
x=0, \quad x=2, \quad x=-3
\end{gathered}
$$

(c) Solve for $x$.

$$
\begin{gathered}
\frac{2}{x}+\frac{1}{3}=4 \\
3 x\left(\frac{2}{x}+\frac{1}{3}\right)=4 \cdot 3 x \\
6+x=12 x \\
6=11 x \\
x=\frac{6}{11}
\end{gathered}
$$

(d) Perform the indicated operation.

$$
\begin{gathered}
\frac{2}{x}+\frac{1}{3} \\
2 \cdot 3+1 \cdot x \\
6+x
\end{gathered}
$$

Solution.
(a) Equal signs are missing between the steps:

$$
\begin{aligned}
& 2 x^{3}+2 x^{2}-12 x \\
& =2 x\left(x^{2}+x-6\right) \\
& =2 x(x-2)(x+3)
\end{aligned}
$$

(b) The first and second lines should end in $=0$.
(c) Nothing is missing.
(d) It looks like the fractions are cleared, which is not possible to do. The denominator is missing and so are equal signs:

$$
\begin{aligned}
& \frac{2}{x}+\frac{1}{3} \\
& =\frac{2 \cdot 3}{3 x}+\frac{1 \cdot x}{3 x} \\
& =\frac{6+x}{3 x}
\end{aligned}
$$

## EXERCISES 3.1

1. What is given below, an equation or an expression?
(a) $y^{4}-4 y^{2}+1$
(b) $\frac{4-x}{x}=1$
(c) $2 s-7=0$
(d) $2 s-7$
(e) $x(3 x+5)=4 x+9$
(f) $x(3 x+5)+4 x+9$
2. Which of the following is an acceptable answer when solving an equation for a real variable? Sets given below represent solution sets.
(a) $\frac{x}{x+1}$
(b) $x=-25, x=8, x=3$
(c) $\{0\}$
(d) $x= \pm \sqrt{2}$
(e) $x=\emptyset$
(f) $x \in\{0,1,3\}$
(g) all reals
3. For each problem given below, solution steps are provided, but some equal signs, numbers, or parts of expressions may be omitted. Insert what is missing.
(a) Solve for x .

$$
\begin{gathered}
2 x-4(5-x)=0 \\
2 x-20+4 x \\
6 x-20 \\
6 x=20 \\
x=\frac{20}{6} \frac{10}{3}
\end{gathered}
$$

(b) Simplify by removing parentheses and combining like terms.

$$
\begin{gathered}
2 x-4(5-x) \\
2 x-20+4 x \\
6 x-20
\end{gathered}
$$

(c) Expand and simplify.

$$
\begin{gathered}
\frac{1}{2} x\left(8 x-\frac{2}{3}\right) \\
\frac{1}{2} x \cdot 8 x-\frac{1}{2} x \cdot \frac{2}{3} \\
4 x^{2}-\frac{1}{3} x
\end{gathered}
$$

(d) Solve for $x$.

$$
\begin{gathered}
\frac{1}{2} x\left(8 x-\frac{2}{3}\right)=0 \\
x\left(8 x-\frac{2}{3}\right)=0 \\
x=0, \quad 8 x-\frac{2}{3}=0 \\
3 \cdot 8 x-3 \cdot \frac{2}{3} \\
24 x-2 \\
24 x=2 \\
x=\frac{1}{12}
\end{gathered}
$$

### 3.2 Types of Equations

Different equations require different solution methods. This is why it is important to recognize different types of equations. The two main types of equations are linear and non-linear equations.

A linear equation has linear expressions on its sides. We have talked about linear expressions in Section 2.2. In a linear equation the variable may be multiplied by a constant and this term may be added to other terms, either of the same kind, or constant. These are the only operations that may be done with the variable. Thus, the variable, say $x$, in a linear equation may be only raised to the first power. If $x$ is raised to a higher power, to a negative number, or a fraction which does not reduce to 1 , then the equation is non-linear. This implies that in a linear equation, $x$ cannot be under a fraction bar or a radical. It cannot be in the exponent either. In other words, a linear equation is either immediately of the form $a x+b=0$, where $a$ and $b$ are some constants and $a \neq 0$, or it is equivalent to this form.

Example 1. Determine for each equation whether it is linear or not.
(a) $6 x-1=3(4+x)$
(b) $5-2 x+3 x^{2}=0$
(c) $\frac{x}{x+1}=7$
(d) $\sqrt{x+6}=2 x-1$
(e) $5 x+\sqrt{2}=9$
(f) $\frac{7 x+3}{4}=2 x+\frac{1}{8}$
(g) $2^{x}-11=5 x+1$

Solution. Linear equations are (a), (e), and (f). Equation (b) is not linear since the highest power of $x$ is 2 and (c) is non-linear because $x$ is under the fraction bar. In (d), $x$ is under a radical, so this equation is not linear either. Note that fractions and radicals are permitted in linear equations, as long as fractions have constant denominators (like in (f)) and radicals have constant radicands (like in (e)). Finally, (g) is non-linear because $x$ is in the
exponent.
Non-linear equations are classified depending on the expressions they have on their sides. If both sides of an equation are polynomials (see Section 2.2), the equation is called a polynomial equation. The degree of a polynomial equation is the highest power of the variable. Linear equations are also polynomial equations - they are first-degree equations. Higher-degree polynomial equations are quadratic (degree $=2$ ), cubic (degree $=3$ ), quartic (degree $=4$ ), etc.

Example 2. Determine for each equation whether it is polynomial or not. If it is, what is its degree?
(a) $\sqrt{6 x-1}=3$
(b) $5-2 x+3 x^{2}=0$
(c) $\frac{x}{x+1}=7$
(d) $7(x+4)=2 x-1$
(e) $5 x^{3}-\frac{2}{9} x^{2}+x-\sqrt{3}=0$
(f) $x^{4}=16$

Solution. Polynomial equations are: (b) with degree 2 (this is a quadratic equation), (d) with degree 1 (linear), (e) with degree 3 (cubic), and (f) with degree 4 (quartic).

Equations like (f) in Example 2 are the simplest polynomial equations. These equations only contain one variable term of the form $a x^{n}$ with $n=1,2,3, \ldots$ and a non-zero constant $a$. When the equation is solved for $x^{n}$, the result is $x^{n}=A$ with another constant $A$. This can be generalized to $x^{p}=A$, where $p$ is any number, but typically a positive rational number. Such equations can be called power equations. When $p$ is a positive integer, a power equation is also a polynomial equation.

Example 3. Determine for each equation whether it is a power equation or not.
(a) $\sqrt{x}=3$
(b) $4 x^{5}+1=0$
(c) $3 x^{2}+x=0$
(d) $x^{2 / 5}=9$
(e) $3^{x}=9$
(f) $x^{4}=16 x$
(g) $x^{3}=9$

Solution. Equations (c) and (f) are not power equations because each contains two variable terms. Equation (e) is not a power equation since $x$ is in the exponent of the power $3^{x}$, whereas power equations have $x$ is in the base of the power (compare (e) to (g)). All other given equations are power equations, (a) because it can be written as $x^{1 / 2}=3$.

Of other non-polynomial equations, let us consider here rational, irrational, and exponential equations. If at least one side of an equation is a rational expression, the equation itself is rational. Irrational equations are defined analogously (for rational and irrational expressions, see Section 2.2). Exponential equations contain a power with the variable in the exponent. Equation (e) in Example 3 illustrates an exponential equation.

Example 4. Classify each equation as polynomial, rational, irrational, or exponential.
(a) $x^{2}+1=x+5^{x}$
(b) $\frac{x}{x+1}=7$
(c) $\left(\frac{2}{3}\right)^{x^{2}-x+1}=1$
(d) $7\left(x^{3}+4\right)=2 x-1$
(e) $2 x-3=x+\sqrt{x}$
(f) $\frac{1}{3} x^{2}=\frac{7}{2}+x$
(g) $\sqrt[3]{6 x+1}=4$

Solution. Polynomial equations: (d) and (f); rational equations: (b); irrational equations: (e) and (g); exponential equations: (a) and (c).

It is important to point out that equations are classified before any solution steps are taken. For instance, if we multiply both sides of

$$
\begin{equation*}
\frac{x^{2}}{x-1}=\frac{1}{x-1} \tag{1}
\end{equation*}
$$

by $x-1$, we get a quadratic equation, $x^{2}=1$. If, because of this, we say that (1) is quadratic, we are wrong. Equation (1) is not equivalent to $x^{2}=1$ since this quadratic equation has two solutions, $x= \pm 1$, and equation (1) has only one solution, $x=-1$. This is why we can only classify (1) as a rational equation.

## EXERCISES 3.2

1. Determine for each equation whether it is linear or not.
(a) $3(x-1)^{2}=5 x-7$
(b) $3(x-1)=5 x-7$
(c) $12+\sqrt{x}=20$
(d) $\sqrt{12}+x=20$
(e) $8 x+9=3^{x}+4$
(f) $\frac{3}{x}=2 x+1$
(g) $x+x^{-1}=4$
2. In the following list, identify all equations that are quadratic or cubic.
(a) $x+2 \sqrt[3]{x}-1=0$
(b) $5 x^{3}=2$
(c) $\frac{x}{2}=3 x-7$
(d) $x^{2}-4=0$
(e) $2 x^{4}+3 x^{2}+8=0$
(f) $x-x^{3}=1$
(g) $x^{2}=3 x+6$
3. Determine for each equation whether it is a power equation or not.
(a) $x^{3}=-8$
(b) $4 \sqrt{x}-1=0$
(c) $3 x^{2 / 3}=x$
(d) $x^{2}-7=0$
(e) $5 x^{3}+2 x=0$
(f) $6 x^{5}-1=9$
(g) $4^{x}+x^{4}=1$
4. Classify each equation as polynomial, rational, irrational, or exponential.
(a) $\frac{1}{x^{2}+1}=8 x-2$
(b) $\frac{3}{5} x^{2}-x=7 x-4$
(c) $x+9=\sqrt{x^{2}-3 x+5}$
(d) $2^{x}+3 x-1=0$
(e) $\frac{6}{\sqrt{x}}=1$
(f) $3 x+5=8(2-7 x)$

### 3.3 Solving Equations

More complicated equations are reduced to simpler ones. This is the main principle for solving equations. Linear equations are the simplest because they are easiest to solve. Other simpler equations are quadratic and power equations. When solving equations, our goal is to get to linear, quadratic, or power equations. When this is impossible to do, we have to use some approximate methods - graphing or numerical. For polynomial equations, there are some special theorems that can tell us more about solutions of these equations. The special methods for polynomial equations are not discussed here, nor are graphing and numerical methods.

Different types of equations are reduced to the simpler ones by different methods. For some of these methods, it cannot be guaranteed that they produce
equations equivalent to the original ones. Whenever such a method is used, it is absolutely necessary to verify all the final "solutions". In fact, it is better to refer to those "solutions" as candidates for solutions, since it may happen that some of them do not satisfy the original equation. Such candidates have to be discarded. On the other hand, when the method used is known to produce equations equivalent to the original ones, the verification step is not necessary (although it is always a good idea to verify all solutions). If all the steps of the solution process are done correctly, the resulting numbers have to satisfy the original equation.

When both sides of an equation are multiplied or divided by a nonzero quantity, the resulting equation is equivalent to the previous one. This may be used to simplify equations. Non-zero quantities include non-zero constants and expressions which are never equal to 0 . After multiplying or dividing an equation by an expression which is equal to 0 for some value(s) of the variable, it cannot be guaranteed that the resulting equation is equivalent to the previous one.

Example 1. For each pair of equations, determine whether they are equivalent or not.
(a) $x^{4}=x^{2}$ and $x^{2}=1$
(b) $4 x^{2}=8 x$ and $x^{2}=2 x$
(c) $\frac{x}{2}+\frac{2}{3}=0$ and $3 x+4=6$
(d) $3(x-5)(x+4)=0$ and $(x-5)(x+4)=0$
(e) $21 x^{2}-9 x+6=0$ and $7 x^{2}-3 x+2=0$
(f) $\left(x^{2}+1\right)(2 x-3)=0$ and $2 x-3=0$

Solution. Equations in (a) and (c) are the only pairs of equations that are not equivalent. Both sides of the first equation in (a) are divided by $x^{2}$ to get the other equation. This in general does not guarantee that the equations are equivalent. In this case, they indeed are not equivalent because $x=0$ solves the first equation but not the second one. Only non-zero quantities can be divided out of an equation: 4 out of the first equation under (b), 3 out of the first equations in (d) and (e), and even the whole expression $x^{2}+1$ (which is always positive) out of the first equation in (f). The first equation
under (c) is equivalent to $3 x+4=0$, not to $3 x+4=6$. In this problem, $0 \cdot 6=0$ is mistaken for $0+6=6$.

One way to get simpler equations is to use factoring. This method is used very often. It always produces equivalent equations, thus the final solutions do not have to be verified. As an example, let us solve the equation $3 x^{2}+4 x=4$ by factoring:

$$
\begin{gathered}
3 x^{2}+4 x=4 \\
3 x^{2}+4 x-4=0 \\
(3 x-2)(x+2)=0 \\
3 x-2=0, \quad x+2=0 \\
x=\frac{2}{3}, \quad x=-2
\end{gathered}
$$

This quadratic equation is reduced by factoring to two linear equations, $3 x-$ $2=0$ and $x+2=0$, which are easy to solve. The first step is very important here. If we are to solve an equation by factoring, we have to transform it so that one of its sides is 0 . It is pointless to start factoring like this:

$$
x(3 x+4)=4 .
$$

The above transformation is correct but what can we conclude from it? There are infinitely many choices of two numbers whose product is $4(2 \cdot 2=4$, $4 \cdot 1=4,-2(-2)=4,-1(-4)=4, \frac{1}{2} \cdot 8=4, \sqrt{2} \cdot \sqrt{8}=4$, etc., etc.). On the other hand, if the product is equal to 0 , at least one of the factors has to be 0 (this is known as the Zero-factor Theorem). Therefore, in order to get all possible solutions, we set each factor equal to 0 .

Example 2. For each equation, decide whether it is ready to be solved by factoring or whether there are some other steps that should be done before starting to factor.
(a) $2 x^{2}-10 x+8=0$
(b) $4 x^{3}=x$
(c) $2 x^{2}+7 x=-3$
(d) $x(x+1)-3(x+1)=0$
(e) $(x+5)(2 x-7)=1$
(f) $x(x-7)+2(x+7)=0$

Solution. Equations (a) and (d) are ready for factoring and the others are not. In (b), (c), and (e), the right-hand-side terms have to be moved so that we get 0 on that side. Besides, the left-hand side needs to be foiled in (e). The same is true of ( f ) - when the left side is simplified, the equation is prepared for factoring. There is no need for simplifying the left side in (d) because $(x+1)$ can be factored out.

After factoring the left side of the above equation (a), we get

$$
2(x-1)(x-4)=0 .
$$

Note that the first factor is constant. Following the general procedure, we may set it equal to 0 , but then we immediately realize that $2=0$ is impossible and we find the solutions from $x-1=0$ and $x-4=0$. Another possibility here is to divide both sides of the factored equation by 2 . Or, we can divide each term on both sides of the original equation by 2 in order to get $x^{2}-$ $5 x+4=0$ and then $(x-1)(x-4)=0$.

Factoring is not a universal method. This means that some equations may have solutions which cannot be found by factoring. For higher-degree polynomial equations, the only universal method that is used in practice is the quadratic formula, which only applies to quadratic equations. The quadratic formula for solving the quadratic equation in standard form,

$$
a x^{2}+b x+c=0,
$$

is

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

which students frequently write incorrectly as

$$
x=-b \pm \frac{\sqrt{b^{2}-4 a c}}{2 a} \text { WRONG! }
$$

If a quadratic equation cannot be solved by factoring, it is wrong to conclude that it has no solution. The quadratic formula has to be used and only the quadratic formula can tell us that the quadratic equation has no solution. In order to discuss the existence of the solution of a quadratic equation, we do
not need the whole quadratic formula - it is sufficient to find the sign of the discriminant $d=b^{2}-4 a c$ (the discriminant is the quantity under the radical in the quadratic formula). If $d$ is negative, the quadratic equation has no (real) solution; otherwise it has solutions which may be impossible to find by factoring.

Example 3. No quadratic equation given below can be solved by factoring. For which of them can we say that there is no solution?
(a) $x^{2}-3 x-2=0$
(b) $4 x^{2}=x+2$
(c) $-x^{2}+3 x=-6$
(d) $2 x^{2}-5=0$
(e) $x^{2}+x+1=0$

Solution. We have to find the discriminant for each equation. Equations (b) and (c) have to be rewritten in standard form $a x^{2}+b x+c=0$ in order to identify the constants $a, b$, and $c$ correctly.
(a) Since $a=1, b=-3$, and $c=-2$, the discriminant is positive: $d=$ $(-3)^{2}-4 \cdot 1 \cdot(-2)=9+8=17$. Therefore, this equation has solutions. A typical mistake is to write $d=-3^{2}-4 \cdot 1 \cdot(-2)=-9+8=-1$. This would lead us into the wrong conclusion that there is no solution. The term $b^{2}$ cannot be negative; if $b$ is negative, the square eliminates the minus.
(b) $a=4, b=-1$, and $c=-2$, therefore, $d=(-1)^{2}-4 \cdot 4 \cdot(-2)=1+32=$ 33. This equation has solutions.
(c) The discriminant is negative: $d=3^{2}-4 \cdot(-1) \cdot(-6)=9-24=-15$. There is no solution.
(d) $d=0^{2}-4 \cdot 2 \cdot(-5)=40$ and solutions exist.
(e) No solution since $d=1^{2}-4 \cdot 1 \cdot 1=-3$.

There also are general formulas for solving polynomial equations of the third and fourth degrees but they are very complicated and impractical. For polynomial equations of degree 5 and higher, such general formulas are impossible to construct. Therefore, factoring remains the main method for solving cubic polynomial equations and those of higher degree.

Factoring may produce non-linear prime factors. For instance when solving the equation $x^{3}-x^{2}-5 x=0$ by factoring, we get $x\left(x^{2}-x-5\right)=0$. The factor $x^{2}-x-5$ is not linear and it cannot be factored further (we only factor using integers). Therefore, the equation $x^{2}-x-5=0$ has to be solved by quadratic formula. Very often, the non-linear prime factor is of the form $a x^{p}+b$, which means that a power equation has to be solved.

When solving a power equation, first solve for the power of the variable, i.e. for $x^{p}$.

Example 4. Solve each equation for the power of the variable.
(a) $3 x^{2}-7=0$
(b) $1-5 x^{3}=0$
(c) $x^{2 / 3}=10$
(d) $9+x^{2}=0$
(e) $x^{5 / 2}+2=0$

Solution. Equation (c) is already solved for the power. In the remaining equations, we get
(a) $x^{2}=\frac{7}{3}$
(b) $x^{3}=\frac{1}{5}$
(d) $x^{2}=-9$
(e) $x^{5 / 2}=-2$

After the power of the variable is by itself on the left side, the power equation in general looks like

$$
x^{n / m}=A,
$$

where $A$ is a number and $m$ and $n$ are non-zero integers such that the fraction $\frac{n}{m}$ is reduced. This rational exponent is then removed by raising each side of the equation to $\frac{m}{n}$. The result is the solution

$$
x=\left\{\begin{aligned}
A^{m / n} & \text { if } n \text { is odd } \\
\pm A^{m / n} & \text { if } n \text { is even }
\end{aligned}\right.
$$

(when $n$ is even and $A<0$, the solution is imaginary). As an illustration, consider the power equations $x^{2}=4$ and $x^{3}=64$. The solutions are respectively

$$
x= \pm 4^{1 / 2}= \pm \sqrt{4}= \pm 2
$$

and

$$
x=64^{1 / 3}=\sqrt[3]{64}=4
$$

This shows why $\pm$ has to be used when $n$ is even. The equation $x^{2}=4$ indeed has two solutions since both $2^{2}=4$ and $(-2)^{2}=4$. Without $\pm$ we would only get $x=\sqrt{4}=2$ and, therefore, we would miss one solution. When taking an even root of a positive number, we only get a positive number. $\sqrt{4}$ is just 2 and never -2 . Since in this problem we need this other, negative, solution, we have to use $\pm$ in front of the square root. As for the other equation, $x^{3}=64,-4$ is not its solution since $(-4)^{3}=-64$. Therefore, $\pm$ is not needed in front of odd roots.

Example 5. The symbol $\pm$ may be either omitted or incorrectly used below. Correct all mistakes.
(a) $x^{2}=7 \Leftrightarrow x=\sqrt{7}$
(b) $\sqrt{81}= \pm 9$
(c) $\sqrt[3]{-8}= \pm 2$
(d) $x^{5}=8 \Leftrightarrow x=\sqrt[5]{8}$
(e) $x^{4}=2 \Leftrightarrow x= \pm \sqrt[4]{2}$
(f) $x^{3}=-9 \Leftrightarrow x= \pm \sqrt[3]{9}$
(g) $\sqrt[4]{16}=2$

Solution. There is nothing to correct in (d), (e), and (g). The answer in (b) is just 9 ; in (c), it is -2 ; and in (f), $-\sqrt[3]{9}$. Finally, $\pm$ should be inserted before $\sqrt{7}$ in (a).

Let us now say a few words about strategies for solving non-polynomial equations. We try to transform them to polynomial equations, which we are more familiar with.

Rational equations are reduced to polynomial equations by clearing fractions. Each term on each side of the equation should be multiplied by the $L C D$ of all fractions in the equation. For instance, the rational equation

$$
\begin{equation*}
2+\frac{1}{x+1}=x \tag{1}
\end{equation*}
$$

can be multiplied through by $(x+1)$ to ultimately get a quadratic equation:

$$
\begin{gather*}
2(x+1)+\frac{1}{x+1}(x+1)=x(x+1)  \tag{2}\\
2 x+2+1=x^{2}+x \\
x^{2}-x-3=0
\end{gather*}
$$

The following is a typical mistake when clearing fractions from rational equations: the multiplication of both sides of equation (1) by $(x+1)$ is written as

$$
\begin{equation*}
(x+1)\left[2+\frac{1}{x+1}\right]=x(x+1) \tag{3}
\end{equation*}
$$

and then, $(x+1)$ is crossed out on the left side of the equation. This gives

$$
2+1=x(x+1) \text { WRONG! }
$$

This is wrong because $(x+1)$ has to be distributed over 2 and $\frac{1}{x+1}$ before the fraction can be cleared. This is why approach (2) is better: the LCD $(x+1)$ is immediate distributed if we write it by each term on each side of the equation.

Example 6. After finding the LCD for all fractions in the given equation, write it by each term of each side of the equation and eliminate the fractions.
(a) $\frac{2}{x^{2}+1}-1=x$
(b) $\frac{2}{x}+x=\frac{1}{x^{2}}$
(c) $3 x-\frac{1}{2 x}=\frac{x+3}{4(x-1)}+5$

## Solution.

(a) $\mathrm{LCD}=x^{2}+1$

$$
\begin{gathered}
\frac{2}{x^{2}+1}\left(x^{2}+1\right)-1\left(x^{2}+1\right)=x\left(x^{2}+1\right) \\
2-x^{2}-1=x^{3}+x
\end{gathered}
$$

(b) $\mathrm{LCD}=x^{2}$

$$
\begin{aligned}
\frac{2}{x} \cdot x^{2}+x \cdot x^{2} & =\frac{1}{x^{2}} \cdot x^{2} \\
2 x+x^{3} & =1
\end{aligned}
$$

(c) $\mathrm{LCD}=4 x(x-1)$

$$
\begin{aligned}
& 3 x \cdot 4 x(x-1)-\frac{1}{2 x} \cdot 4 x(x-1) \\
& \quad=\frac{x+3}{4(x-1)} \cdot 4 x(x-1)+5 \cdot 4 x(x-1) \\
& 12 x^{2}(x-1)-2(x-1)=(x+3) x+20 x(x-1)
\end{aligned}
$$

There is another method for solving rational equations, which keeps the fractions until the very last step. This method is highly recommended because it also has to be used when solving rational inequalities, as we shall see soon in Section 4.3. The following steps should be taken:

1. bring all terms to the same side of the equation, leaving 0 on the other side,
2. transform the non-zero side to a single fraction by performing all operations indicated on that side,
3. set the numerator (top) of the fraction equal to 0 - this results in a polynomial equation.

Let us illustrate this approach by applying it to equation (1):

$$
\begin{gathered}
2+\frac{1}{x+1}=x \\
2-x+\frac{1}{x+1}=0 \\
\frac{(2-x)(x+1)}{x+1}+\frac{1}{x+1}=0 \\
\frac{2 x+2-x^{2}-x+1}{x+1}=0 \\
-x^{2}+x+3=0
\end{gathered}
$$

The switch from a rational equation to a polynomial one in the last step is based on the principle that a fraction is equal to 0 if and only if its numerator is equal to 0 . This is so because both sides of the equation $\frac{p}{q}=0$ should be multiplied by the denominator $q$ to get the numerator $p=0$.

Example 7. Transform each equation to an equivalent equation of the form fraction $=0$.
(a) $\frac{2}{x^{2}+1}-1=x$
(b) $\frac{2}{x}+x=\frac{1}{x^{2}}$
(c) $3 x-\frac{1}{2 x}=\frac{x+3}{4(x-1)}+5$

Solution.
(a) $\frac{2}{x^{2}+1}-x-1=0 \Leftrightarrow \frac{2}{x^{2}+1}-\frac{(x+1)\left(x^{2}+1\right)}{x^{2}+1}=0$

$$
\Leftrightarrow \quad \frac{2-(x+1)\left(x^{2}+1\right)}{x^{2}+1}=0
$$

(b) $\frac{2}{x}+x-\frac{1}{x^{2}}=0 \Leftrightarrow \frac{2 x+x^{3}-1}{x^{2}}=0$
(c) $3 x-5-\frac{1}{2 x}-\frac{x+3}{4(x-1)}=0$
$\Leftrightarrow \quad \frac{(3 x-5) \cdot 4 x(x-1)-2(x-1)-(x+3) x}{4 x(x-1)}=0$

When a rational equation is solved it is absolutely necessary to check the candidates for solutions. When clearing fractions from a rational equation, we multiply the equation, by the LCD, which may be equal to 0 for some values of the variable. Recall that this may result in a an equation which is not equivalent to the original one. Therefore, either method for solving rational equations may produce some values of the variable, which do not solve the originally given equation. For instance, the equation

$$
\frac{2}{x^{2}-1}+1=\frac{1}{x-1}
$$

reduces to the polynomial equation

$$
x^{2}-x=0,
$$

whose solutions are $x=0$ and $x=1$. However, only $x=0$ can be accepted as a solution of the original equation. Whereas $x=1$ is indeed a solution of the polynomial equation, it does not solve the rational one since it makes the fractions undefined. We got this "phantom solution" not because we did something wrong, but because we had to use methods which may yield non-equivalent equations and superfluous solutions. Therefore, the numbers we get when solving a rational equation are just candidates for solutions and those candidates should be verified. If we are sure that our work is correct, we should just discard all numbers that make some fractions of the original equation undefined, that is, some denominators equal to 0 .

The first step when solving irrational equations is to isolate the radical on one side. We are discussing here those irrational equations that only have one radical with the variable underneath. After isolating the radical, both sides of the equation are raised to the appropriate power which will cancel the radical. This power is the same as the index of the radical. Consider, for instance,

$$
\begin{equation*}
3-2 \sqrt{x}=x . \tag{4}
\end{equation*}
$$

We can start solving this equation by subtracting 3 from both sides of the equation. This gives

$$
\begin{equation*}
-2 \sqrt{x}=x-3 \tag{5}
\end{equation*}
$$

Since the radical term is the only term on the left side of the equation, this concludes the step in which we isolate the radical. Note that this does not mean that the equation has to be solved for the radical. If we wanted to solve (5) for $\sqrt{x}$, we would have to divide both sides by -2 , which would give us an unpleasant fraction on the right side. This is unnecessary. We can square both sides of (5) to eliminate the square root and obtain a quadratic equation which can be solved by factoring:

$$
\begin{gathered}
(-2 \sqrt{x})^{2}=(x-3)^{2} \\
4 x=x^{2}-6 x+9 \\
x^{2}-10 x+9=0 \\
(x-9)(x-1)=0
\end{gathered}
$$

Note that on the left side we have to square both the radical and its coefficient. The resulting quadratic equation has two solutions: $x=9$ and $x=1$. However, $x=9$ has to be discarded since it does not solve the original equation (4). We conclude that $x=1$ is the only solution of (4). This illustrates that radical equations are another class of equations for which it is absolutely necessary to check the "solutions". Raising both sides of an equation to the same power does not necessarily produce equivalent equations.

Forgetting the first step is a typical mistake when solving rational equations. In problem (5), this mistake would give

$$
\begin{equation*}
(3-2 \sqrt{x})^{2}=x^{2}, \tag{6}
\end{equation*}
$$

which, actually, is not wrong, but cannot eliminate the radical. The left side of (6) should be foiled properly and the resulting equation still contains a radical:

$$
9-12 \sqrt{x}+4 x=x^{2}
$$

The main problem is that (6) is usually combined with another mistake that of applying an exponent term by term:

$$
9+4 x=x^{2} \text { or even } 9-4 x=x^{2} \quad \text { WRONG! }
$$

This happens if we square 3 and $2 \sqrt{x}$ separately, without foiling the left side of (6).

Example 8. For each equation, decide whether it is prepared for the elimination of the radical. If it is not, do the necessary steps.
(a) $2 x-1=3 \sqrt{x+4}$
(b) $x+\sqrt{x}=5$
(c) $\sqrt[3]{1-x}=2$
(d) $x^{2}+3 \sqrt{x}-4=0$
(e) $\sqrt[3]{2 x+7}+2=x$

Solution. Equations (a) and (c) are the only ones that have radicals isolated on one side. They are, therefore, prepared for radical elimination by squaring both side of (a) and by cubing both sides of (c). Equation (b) should be transformed to $\sqrt{x}=5-x$. Similarly, equations (d) and (e) should be rewritten as $3 \sqrt{x}=4-x^{2}$ and $\sqrt[3]{2 x+7}=x-2$ respectively.

## The simplest exponential equations are of the form

$$
\begin{equation*}
A \cdot a^{b x+c}+B=0, \tag{7}
\end{equation*}
$$

where $A, B, a, b$, and $c$ are given constants, such that $A \neq 0, a>0, a \neq 1$, and $b \neq 0$. When solving exponential equations of type (7), the first step is to solve it for the exponential expression $a^{b x+c}$. After this, the general step is either to switch to logarithmic notation or to apply the appropriate logarithm to both sides of the equation. This method can never produce unacceptable answers, so it is not necessary to check the solution in this case.

Example 9. Solve each equation for the exponential expression of the form $a^{b x+c}$.
(a) $3^{1-x}+5=20$
(b) $5+2^{3 x-4}=8$
(c) $1-6^{x}=3$
(d) $4-2 \cdot 5^{x+7}=0$
(e) $7+3 \cdot\left(\frac{2}{5}\right)^{2 x-1}=14$

## Solution.

(a) $3^{1-x}=15$
(b) $2^{3 x-4}=3$
(c) $6^{x}=2$
(d) $5^{x+7}=2$
(e) $\left(\frac{2}{5}\right)^{2 x-1}=\frac{7}{3}$

Example 10. For each equation, decide whether it is a must or not to check its solutions.
(a) $\sqrt{2 x-3}=x+1$
(b) $x^{4}=2 x^{2}$
(c) $1-6^{x}=3$
(d) $\frac{x+6}{x-1}+3=\frac{1}{x}$
(e) $3(2-5 x)+9 x=x-2(7-x)$
(f) $5+2 \sqrt{x}=7 x$

Solution. Only rational and irrational equations require that solutions must be checked. Therefore, solutions have to be checked in (a), (d), and (f). Equation (b) is polynomial, (c) is exponential, and (e) is linear. If we solve them correctly, each solution has to be acceptable.

Some equations have no solution. For instance, no real number $x$ satisfies the equation $x^{2}=-4$. This equation has no real solutions, only imaginary ones. There are also equations that do not have solutions in any set of numbers, e.g. $x+1=x+7$. As we have already seen in Section 3.1, when an equation is without solutions, the answer should be "no solution", or $\emptyset$.

Example 11. Which of the following is an acceptable answer when solving an equation which has no real solution?
(a) $x=0$
(b) no solution
(c) $x=\emptyset$
(d) $\{\emptyset\}$
(e) 0
(f) $\emptyset$

Solution. Only (b) and (f) are acceptable answers. In (a) and (e), 0 is a number and it does not mean the empty set. In (c), we have an equality of a number and a set, which is impossible. Finally, (d) means that we solved an equation whose unknown is not a real variable but a set. This, for example, would be the answer when solving the following problem (see Section 1.1): find the set $X$ such that $X \cap X=\emptyset$. Since the solution is $X=\emptyset$, the solution set for this problem is $\{\emptyset\}$.

Some equations have all real numbers for their solutions. The following is an example:

$$
\begin{equation*}
3 x-2(x+1)=4(x-1)-3 x+2 \tag{8}
\end{equation*}
$$

When each side of this linear equation is simplified, the equation becomes $x-2=x-2$, which is an identity. Identities are equalities that are true for all real numbers, or at least for all real numbers that make all expressions in the equality defined. More specifically, the word identity is used for useful general formulas like the difference of two squares,

$$
a^{2}-b^{2}=(a-b)(a+b),
$$

or for equalities like

$$
\begin{equation*}
(x-1)^{2}=x^{2}-2 x+1 \tag{9}
\end{equation*}
$$

for which we can say at once that they are true for all real numbers. No one would say immediately that (8) was an identity - we discover that this equation reduces to an identity after we do some work. On the other hand, we may be given (9) as an equation to be solved for $x$. If we immediately realize that this is an identity, we just say that the solution is all real numbers, or that the solution set is $\mathbb{R}$.

Example 12. Determine for each equation given below whether its solution set is $\mathbb{R}$ or not.
(a) $x^{2}-3 x+2=0$
(b) $5 t+4=4+5 t$
(c) $\sqrt{x^{2}+1}=x+1$
(d) $x^{2}-3 x+2=(x-1)(x-2)$
(e) $\frac{s+1}{s}=1+\frac{1}{s}$

Solution Each of the equations (b), (d), and (e) is either an identity or it reduces to one. Whereas (b) and (d) are satisfied for all real numbers, (e) is not. Equation (e) is true for all values of $s$ for which both sided are defined, that is for all $s \neq 0$. Equations (a) and (c) are not identities either, the latter because radicals cannot be applied term by term (see Section 2.3). In conclusion, $\mathbb{R}$ is the solution set only for (b) and (d).

Example 13. Which of the following is an acceptable answer when solving an equation whose solution is any real number?
(a) $\mathbb{R}$
(b) $\{\mathbb{R}\}$
(c) all real numbers
(d) $x=\mathbb{R}$
(e) $x \in \mathbb{R}$

Solution. Only answers (b) and (d) are unacceptable. Answer (b) cannot be accepted for the same reason as in Example 11 (d), and (d) because, again, numbers and sets cannot be equal.

## EXERCISES 3.3

1. For each pair of equations, determine whether they are equivalent or not.
(a) $6 x^{4}-8 x=0$ and $3 x^{4}-4 x=0$
(b) $10 x^{2}=5 x$ and $2 x=1$
(c) $\frac{1}{2} x^{2}-5 x+9=0$ and $x^{2}-5 x+9=0$
(d) $2\left(x^{2}-5\right)(x+4)=0$ and $x+4=0$
(e) $-x^{2}+3 x-6=0$ and $x^{2}-3 x+6=0$
(f) $\frac{2 x}{7}+\frac{1}{14}=0$ and $4 x+1=0$
2. For each equation, decide whether it is ready to be solved by factoring or whether there are some other steps that should be done before starting to factor.
(a) $x^{2}-4=3 x$
(b) $x^{2}-9=0$
(c) $5\left(x^{2}+1\right)-x\left(x^{2}+1\right)=0$
(d) $7 x^{2}=2 x$
(e) $(x-3)(x+2)+6=0$
(f) $3 x^{3}+4 x^{2}-4 x=0$
3. No quadratic equation given below can be solved by factoring. For which of them can we say that there is no real solution?
(a) $x^{2}=x-5$
(b) $-2 x^{2}+4 x-1=0$
(c) $x(x-6)=3$
(d) $3 x^{2}-x+2=0$
(e) $7 x^{2}+9=0$
4. Solve each equation for the power of the variable.
(a) $2 x^{5}+9=0$
(b) $4 x^{1 / 3}-1=0$
(c) $5-x^{2}=0$
(d) $7+2 x^{3}=0$
(e) $x^{3 / 2}=4$
5. The symbol $\pm$ may be either omitted or incorrectly used below. Correct all mistakes.
(a) $x^{3}=7 \Leftrightarrow x=\sqrt[3]{7}$
(b) $\sqrt[4]{81}= \pm 3$
(c) $x^{2}=6 \Leftrightarrow x=\sqrt{6}$
(d) $x^{6}=2 \Leftrightarrow x= \pm \sqrt[6]{2}$
(e) $\sqrt[5]{32}= \pm 2$
(f) $\sqrt{36}= \pm 6$
(g) $x^{3}=27 \Leftrightarrow x= \pm 3$
6. After finding the LCD for all fractions in the given equation, write it by each term of each side of the equation and eliminate the fractions.
(a) $2 x+\frac{1-x}{4 x+3}=6$
(b) $x^{2}-\frac{x+2}{6 x}=\frac{1}{8 x}$
(c) $x+4-\frac{1}{x+1}=\frac{x}{x+3}$
7. Transform each equation to an equivalent equation of the form fraction $=0$.
(a) $2 x+\frac{1-x}{4 x+3}=6$
(b) $x^{2}-\frac{x+2}{6 x}=\frac{1}{8 x}$
(c) $x+4-\frac{1}{x+1}=\frac{x}{x+3}$
8. For each equation, decide whether it is prepared for the elimination of the radical. If it is not, do the necessary steps.
(a) $2 \sqrt{x+5}-1=11$
(b) $-6 \sqrt{x^{2}+1}=5$
(c) $x+7=\sqrt[3]{x}$
(d) $2-x=\sqrt{x}+9$
(e) $x+\sqrt[3]{x}=0$
9. Solve each equation for the exponential expression of the form $a^{b x+c}$.
(a) $2 \cdot 3^{2 x+4}-10=0$
(b) $2^{1-2 x}-7=0$
(c) $9-2 \cdot 3^{x}=1$
(d) $4+3 \cdot 5^{x-4}=12$
(e) $8-\frac{1}{2} \cdot\left(\frac{3}{7}\right)^{x}=5$
10. For each equation, decide whether it is a must or not to check its solutions.
(a) $\frac{2 x-3}{2}=x+1$
(b) $\frac{1}{x+1}=\frac{x}{x-1}+3$
(c) $x^{2}-3 x=3$
(d) $\sqrt{3 x+1}=x+1$
(e) $7-3 \cdot 2^{7 x-4}=10$
11. Determine for each equation given below whether its solution set is $\mathbb{R}$ or not.
(a) $3 x-5=7 x+4$
(b) $5(y+4)=5 y+20$
(c) $\frac{x}{x^{2}+1}=\frac{1}{x+1}$
(d) $(2 u-1)^{2}=4 u^{2}-4 u+1$
(e) $\frac{v^{2}+v}{v+1}=v$
12. Which of the following is an acceptable answer when solving an equation whose solution is any real number not equal to 0 ?
(a) $\mathbb{R} \neq 0$
(b) $x \neq 0$
(c) $x>0$
(d) $\{x \mid x$ is a real number and $x \neq 0\}$
(e) $x>0$ or $x<0$
