5 Functions

5.1 Defining Functions

Equations in two variables can help us explain what functions are. The two variables are typically denoted by $x$ and $y$ – they take their values in the set of all real numbers, $\mathbb{R}$. Consider the following two equations:

(1) \[ 6x - 3y = 1 \]

and

(2) \[ 5x + y^2 = 4. \]

Each equation in two variables shows how the variables are related to each other. We can freely choose the value for one of the variables, but the value of the other variable is determined after this. For instance, if we set $x = 3$ in (1), then $y$ has to equal $\frac{17}{3}$. Or, if $y = -2$ in (2), then $x$ has to be equal to 0.

Equations in two variables can be graphed. As we have seen, $x = 0$ and
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$y = -2$ satisfy equation (2) and the point with coordinates $x = 0$ and $y = -2$ can be plotted in the rectangular coordinate system. When we find other pairs of $x$ and $y$ values satisfying (2) and we plot all the corresponding points in the same coordinate system, we get the graph of equation (2). Like the equation itself, the graph also shows how the variables $x$ and $y$ are related. Therefore, the relation between two variables can be expressed by an equation or by a graph. Another possibility is to provide a table of $x$ and $y$ values, which shows how those values are paired.

The simplest equations in two variables are linear. They are of general form $ax + by = c$, where $a$, $b$, and $c$ are given constants. Equation (1) is an example of a linear equation in $x$ and $y$. For a linear equation in $x$ and $y$ it is important that the only operations that can be done with each variable are multiplication by a constant and addition of the resulting term to other terms. This means that both $x$ and $y$ are raised to no other power but first and that $x$ and $y$ are in separate terms. Neither variable can be under a radical or a fraction bar. The constants $a$ and $b$ may not both equal 0 since this would leave the equation without variables at all. However, one of them may be equal to 0. Even equations like $x = 3$ or $y = -1$ may be interpreted in the two-variable context. We can think of them as $x + 0 \cdot y = 3$ and $0 \cdot x + y = -1$ respectively. Thus, the meaning of $x = 3$ is that $x$ equals 3 while $y$ can be any real number. In the same way, $y = -1$ means that $y$ equals $-1$ for any value of $x$. Therefore, equations $x = 3$ and $y = -1$ are linear. The graph of any linear equation in $x$ and $y$ is a straight line. The other way around, if the graph of an equation is a straight line, the equation is linear. Equations like $x = 3$ and $y = -1$ are special and their graphs are special too: the graph of $x = 3$ is a vertical line (all points in the rectangular coordinate system whose $x$ coordinate is 3) and $y = -1$ is a horizontal line (all points in the rectangular coordinate system whose $y$ coordinate is $-1$).

Example 1. Determine without graphing whether the graph of each given equation is a straight line or not.

(a) $3x + \frac{1}{2}y = 0$

(b) $y = 7x - 1$

(c) $y = 7x^2 - 1$

(d) $2x + y = 1$
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(e) \( 2xy = 1 \)
(f) \( x = -4 \)
(g) \( y = \sqrt{x - 5} \)
(h) \( 3y - 4 = 0 \)

Solution. Equations (a), (b), (d), (f), and (h) are linear. They therefore have straight lines for their graphs. In particular, (h) is equivalent to \( y = \frac{4}{3} \) and its graph is a horizontal line. Equations (c), (e), and (g) are not linear because \( x \) is squared in (c), \( x \) and \( y \) are not in separate terms in (e), and \( x \) is under the square root in (g). Graphs of (c), (e), and (g) are some curves, not straight lines.

The two variables can be treated differently. Since equations in \( x \) and \( y \) have one degree of freedom, we can choose \( x \) to be the first variable that real values are assigned to. Such value assignments do not depend on anything and, therefore, \( x \) is called the independent variable. Once \( x \) takes a specific value, it is determined by the equation what the value(s) of \( y \) are. This is why \( y \) is called the dependent variable – its values depend on \( x \). Equations in two variables, as well as graphs or tables, show how \( y \) depends on \( x \). This dependence is understood very generally and it includes the special cases, like in the above-mentioned equation \( y = -1 \), in which \( y \) in fact does not depend on \( x \) (in this example \( y \) is always equal to \(-1\) regardless of \( x \)). In some other cases, there may be \( x \) values for which there is no corresponding \( y \). For instance, if \( x = 1 \) in equation (2), then it follows that \( y^2 = -1 \), which is not possible for any real number \( y \).

Consider all \( x \) values for which there is a value of \( y \). If for each \( x \) value of this kind there is one and only one value of \( y \), then \( y \) is a function of \( x \). When we say this, we indicate that \( y \) depends on \( x \) in a special way: no value of \( x \) can be paired with more than one value of \( y \). This dependence is therefore deterministic. When a value of \( x \) is chosen, the value of \( y \) is determined uniquely (if this \( y \)-value exists). For instance, in equation (1), \( x \) may be any real number and to each choice of \( x \) there corresponds exactly one \( y \)-number. We can see this by solving the equation for \( y \):

\[
y = 2x - \frac{1}{3}.
\]
All operations on the right side of the above equation are defined (we can always multiply and subtract two numbers) and deterministic (multiply or subtract two numbers and you will get one answer).

**Not every equation in** \(x\) **and** \(y\) **describes** \(y\) **as a function of** \(x\). The variable \(y\) may depend on \(x\) in a different, less deterministic way. If we set \(x = 0\) in equation (2), we get two corresponding values of \(y\), \(y = \pm 2\). Here, \(y\) **is not a function of** \(x\). This too can be seen by solving the equation for \(y\):

\[
y = \pm \sqrt{4 - 5x}.
\]

For most \(x\)-values, making the above square root defined, there are two \(y\) values because of the \(\pm\) sign in front of the square root. For most, but not for all – if \(x = \frac{4}{5}\), then \(y = \pm 0 = 0\). It would be very wrong to conclude from looking at the pair \(x = \frac{4}{5}\) and \(y = 0\) that \(y\) is a function of \(x\) in (2). One value of \(x\) giving a unique \(y\)-value means nothing. **Each** \(x\)-value should have no more than one \(y\)-value assigned to it. Therefore, in order to determine whether in a given equation \(y\) is a function of \(x\) or not, the following question has to be asked: **Is there an** \(x\)-value **that gives more than one** \(y\)-value? **If the answer is YES,** \(y\) **IS NOT a function of** \(x\). **If the answer is NO,** \(y\) **IS a function of** \(x\). As we have seen, one way of answering this question is to solve the equation for \(y\), if possible.

Let us consider the equation \(y = x^2\) in order to illustrate one of the typical mistakes when trying to decide whether \(y\) is a function of \(x\) or not. On realizing that \(x = 2\) and \(x = -2\) both produce \(y = 4\), it is sometimes concluded that \(y\) is not a function of \(x\). This is wrong. It is acceptable for a function to have the same \(y\)-value assigned to two or more different \(x\)-values, as long as this is the *only* \(y\)-value assigned to them. Again, what is not acceptable is that there is an \(x\)-value producing two or more \(y\)-values. Even \(y = -1\) is a function of \(x\) because \(y = -1\) is assigned to each real number \(x\), but no \(x\) has two \(y\)-values assigned to it.

**Example 2.** Does the given equation describe \(y\) as a function of \(x\)?

(a) \(x^2 + 3y = 4\)

(b) \(x + y^3 = 1\)

(c) \(xy = 5\)
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(d) \( x^2 + y^4 = 6 \)

(e) \( y = 7|x| - 1 \)

(f) \( |y| = 8 - 3x \)

(g) \( y = \sqrt{2} \)

Solution. Only equations (d) and (f) do not describe \( y \) as a function of \( x \). When (d) is solved for \( y \), the equation \( y = \pm \sqrt{6 - x^2} \) shows that because of \( \pm \) we get two \( y \)-values for each \( x \) which makes the radicand positive. As for (f), if we set \( x = 0 \), we get the equation \( |y| = 8 \), which has two solutions, \( y = \pm 8 \). We can solve all other equations for \( y \) to see that no operation on the right-hand side produces more than one result:

(a) \( y = \frac{4 - x^2}{3} \)

(b) \( y = \frac{\sqrt{1 - x}}{x} \)

(c) \( y = \frac{5}{x} \)

The same is true of equations (e) and (g) which are already solved for \( y \).

Functions can be defined more generally. A function is a rule that assigns to each element \( x \) of a set \( D \) (called the domain) one and only one element \( y \) of set \( R \) (called the range). In order to describe a function, we have to say what the domain is and what the rule is which tells us how \( y \)-values are assigned to \( x \)-values. When the variables \( x \) and \( y \) are real numbers, equations in \( x \) and \( y \) form a special class of rules – those that can be expressed using operations with numbers. In that case, the domain is usually left unspecified on understanding that it only contains the \( x \)-values for which \( y \)-values can be determined. For instance, there is no \( y \)-value that can be assigned to \( x = 0 \) in Example 2(c). When \( x = 0 \), the equation becomes \( 0 = 5 \) which is not only without \( y \), but it is not even true. Or, if we solve the equation for \( y \) and then set \( x = 0 \), we get \( y = \frac{5}{0} \), which is not defined. Either way, \( x = 0 \) is not in the domain of this function. Every other real number \( x \) is in the domain because we can divide 5 by any non-zero number. We shall see in Section 5.4 how to discuss domains in general.

Example 3. Determine for the given number \( x \) whether it is in the domain of the given function \( y \).
(a) \[ x = 3, \ y = \sqrt{5-x} \]

(b) \[ x = \frac{11}{2}, \ y = \sqrt{5-x} \]

(c) \[ x = -3, \ y = \frac{2x - 7}{x^2 - 9} \]

(d) \[ x = 1, \ y = \frac{2x - 7}{x^2 - 9} \]

(e) \[ x = 12, \ y = 7|x| - 1 \]

Solution. The answer is ‘yes’ in (a), (d), and (e), and ‘no’ elsewhere. □

When describing a function, the range does not have to be (but still may be) defined. It is not necessary to define the range because it can be reconstructed from the rule and the domain. To illustrate this, let us use the function \( y = x^2 \) again. The domain here consists of all real numbers. When we square a real number, we cannot get a negative result. Therefore, the range of this function contains all non-negative real numbers.

Functions do not have to be about numbers. Note that the general definition does not mention equations or real numbers. The variables do not have to be real numbers and the rule can be given even verbally (not as an equation, graph, or table). The following example illustrates this. The range is defined in order to make the description more understandable.

Example 4. Is \( y \) a function of \( x \)?

(a) \( D = \) the set of all first names used in the U.S.
   \( R = \) the set of all last names used in the U.S.
   rule: \( x \) is the first name and \( y \) is the last name of the same U.S. resident

(b) \( D = \) the set of all last names used in the U.S.
   \( R = \) the set of all non-negative integers
   rule: \( y \) is the number of children in any family with last name \( x \)

(c) \( D = \) the set of all families in the U.S.
   \( R = \) the set of all non-negative integers
   rule: \( y \) is the number of children family \( x \) has at a particular moment
(d) \( D = \) the set of all students in a particular class  
\( R = \) the set of all letter grades used in this class  
rule: \( y \) is the class grade student \( x \) will earn

(e) \( D = \) the set of all U.S. residents with taxable income  
\( R = \) the set of all non-negative integers  
rule: \( y \) is the amount \( x \) has to pay for federal tax in 2007

\textit{Solution.} The variable \( y \) is a function of \( x \) in (c) (each family has a specific number of children at a particular moment in time), in (d) (each student will get just one grade), and in (e) (each person will pay one tax amount). The rule in (a) does not describe a function because there are many people with the same first name and different last names. Neither is \( y \) a function in (d), because there are many families which have the same last name but a different number of children.

In the remaining sections, we are only going to consider functions that are described by equations and that have some subsets of \( \mathbb{R} \) for their domain and range.

**EXERCISES 5.1**

1. Determine without graphing whether the graph of each given equation is a straight line or not.
   (a) \( x^2 + y^2 = 3 \)
   (b) \( x + y = 3 \)
   (c) \( x + \sqrt{y} = 3 \)
   (d) \( xy = 3 \)
   (e) \( \frac{1}{2}x + 4y = 9 \)
   (f) \( \frac{1}{x} + 4y = 9 \)
   (g) \( \sqrt{5}x - 2y = 1 \)
   (h) \( 2x - 6 = 7 \)

2. Does the given equation describe \( y \) as a function of \( x \)?
   (a) \( 2x + 3|y| = 9 \)
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(b) \(2|x| + 3y = 9\)
(c) \(y = \frac{x^2 - 4}{2}\)
(d) \(x - y^2 = 6\)
(e) \(x - y^3 = 6\)
(f) \(y^2 + x = (y - 1)^2\)
(g) \(x = 7\)

3. Determine for the given number \(x\) whether it is in the domain of the given function \(y\).

(a) \(x = -8, y = \sqrt[3]{2x - 11}\)
(b) \(x = -8, y = \sqrt[3]{2x - 11}\)
(c) \(x = 8, y = \sqrt[3]{2x - 11}\)
(d) \(x = -1, y = \frac{2x}{x^2 + 1}\)
(e) \(x = 0, y = \frac{2x}{x^2 + 1}\)
(f) \(x = 0, y = \frac{x^2 + 1}{x^2 - 2x}\)

4. Is \(y\) a function of \(x\)?

(a) \(D = \) the set of all last names used in the U.S.
    \(R = \) the set of all first names used in the U.S.
    rule: \(x\) is the last name and \(y\) is the first name
    of the same U.S. resident

(b) \(D = \) the set of all U.S. residents
    \(R = \) the set of all last names used in the U.S.
    rule: \(y\) is the last name of resident \(x\) at some particular time

(c) \(D = \) the set of all first names used in the U.S.
    \(R = \) the set of all U.S. residents
    rule: \(x\) is the first name of resident \(y\)

(d) \(D = \) the set of all items that can be bought
    in a particular store at a particular time
    \(R = \) the set of all non-negative numbers
    rounded to two decimal places
    rule: \(y\) is the price paid for item \(x\)
5.2 Function Notation

Different functions can be called differently. We often say that \( y \) is a function, by which we mean that it is a function of \( x \). But, \( y \) is a variable (the dependent one) and a function is a special kind of rule, see the previous section. This may be confusing. In order to distinguish between the dependent variable and the rule itself, the rule, that is, the function, can be named. The same way different objects are referred to by different names, different functions can be named differently. Typical function names are single letters \( f, g, h \), etc., or the corresponding capitals, \( F, G, H \), etc.

In the following two equations, \( y \) is a function of \( x \):

\[
y = 2x - 3, \quad y = x^2.
\]

If we are asked to find \( y \) when \( x \) is equal to, say, 2, we are not sure whether both equations are meant or just one, but we do not know which one. The question is ambiguous but we may answer: “When \( x = 2 \), \( y = 1 \) in the first equation and \( y = 4 \) in the second equation.” This is obviously clumsy. All these problems disappear if function names are used. The two equations can be represented as

\[
f(x) = 2x - 3 \quad \text{and} \quad g(x) = x^2
\]

respectively. Then the above question can be made precise: “What is \( f(2) \), or what is \( g(2) \)?” The answer is simply \( f(2) = 1 \), or \( g(2) = 4 \). Equations given in (2) use function notation. In general, function notation is of the form

\[
f(x) = \text{expression}.
\]

This means that we have a rule (a formula), whose name is \( f \), consisting of all operations given in the expression on the right side, which we apply to the independent variable \( x \). In this notation, the independent variable is separated from the name of the function by parentheses. We read \( f(x) \) as “\( f \) of \( x \)”, which may be viewed as an abbreviation for “the function called \( f \) is
a function of $x$”.

It is important to realize that what defines the function in (3) is the operations on the right side regardless of what the independent variable is. Therefore, the function $f$ in (2) describes the following rule: multiply the independent variable by 2 and subtract 3. Similarly, the function $g$ in (2) is telling us to square the independent variable. The name of the independent variable may change, but these operations remain the same:

$$f(t) = 2t - 3, \quad g(t) = t^2, \quad f(s) = 2s - 3, \quad g(z) = z^2, \quad \text{etc.}$$

When we have to find $f(2)$ and $g(2)$, this means that the independent variable is specified (it is equal to 2) and we have to do the same operations with the specific number. Therefore,

$$f(2) = 2 \cdot 2 - 3 = 1 \quad \text{and} \quad g(2) = 2^2 = 4.$$

We also say in this example that when the function $f$ takes input 2, its output is 1. Likewise, the function $g$ transforms (or maps) input 2 into output 4. In general, $f$ maps input $x$ into output $2x - 3$ and $g$ maps $x$ into $x^2$.

The independent variable $y$, when it is used, denotes the general output of the function. We introduced functions via equations in two variables, typically $x$ and $y$. When function notation is used, $y$ does not have to be written, see (2) and (3). But, if we want to name the output, we usually call it $y$ (or we may use some other letter, different from the independent variable). Therefore, (3) can be written as $f(x) = \text{expression} = y$. However, it is customary to write $y$ first:

(4) \hspace{1cm} y = f(x).

This is by analogy to equations like those in (1). If no specific formula is given for $f(x)$, (4) is the general notation which means that $y$ is a function of $x$. On the other hand, if the function $f$ is specified, we may write (4) as $y = f(x) = \text{expression}$, where the expression indicates what operations should be performed on input $x$ to get output $y$. At the same time, $f$ is the name of the rule that consists of these operations. For instance, we can write $y = f(x) = 2x - 3$ and $y = g(x) = x^2$ for the functions in (2). The only difference between this and (2) is that here the output is named $y$.

Example 1. Identify the name of each function and its input and output.
5.2 Function Notation

(a) \( f(x) = 7x - 1 \)

(b) \( y = h(x) \)

(c) \( F(t) = 6t^2 - t + 2 \)

(d) \( g(u) = z \)

(e) \( G(-3) = 1 \)

(f) \( T = f(R) \)

Solution. From (a) to (f), the names of the functions are \( f, h, F, g, G, \) and \( f \); the inputs are \( x, x, t, u, -3, \) and \( R \); and the corresponding outputs are \( 7x - 1, y, 6t^2 - t + 2, z, 1, \) and \( T \).

Example 2. Write each of the following statements using function notation.

(a) the function is \( f \) and its input and output are \( x \) and \( y \) respectively

(b) the function is \( H \) and its input and output are 0 and \(-5\) respectively

(c) the function is \( k \) and its input and output are \( t \) and \( u \) respectively

(d) the function is \( g \) and its input and output are \( x \) and \( 3x^2 - 1 \) respectively

Solution. (a) \( y = f(x) \) (or \( f(x) = y \)), (b) \( H(0) = -5 \), (c) \( u = k(t) \) (or \( k(t) = u \)), and (d) \( g(x) = 3x^2 - 1 \).

Example 3. The function \( f(x) = 1 - 4x^2 \) is to be evaluated (i.e. its output should be found) for \( x = 0, -1, 2 \). Which of the following answers use correct notation?

(a) \( 0 = 1, -1 = 3, 2 = -15 \)

(b) \( y = 1, y = -3, y = -15 \)

(c) \( f(0) = 1, f(-1) = -3, f(2) = -15 \)

(d) \( f(x) = 1, f(x) = -3, f(x) = -15 \)
Solution. The outputs corresponding to \( x = 0, -1, \) and 2 are indeed 1, -3, and -15 respectively, but only (c) expresses this in a completely correct way. The equalities stated in (a) are not true. This mistake is based on a free interpretation of the equal sign: input 0 produces output 1 is written as \( 0 = 1 \), etc. However, equal sign does not mean this. What we have in (b) is better but is not quite precise – this format would require something like “when \( x = 0, y = 1 \)” etc. Finally, in (d), \( f(x) \) is, strictly speaking, none of the given numbers, but \( 1 - 4x^2 \).

Function notation is reserved for functions. If we write \( y = f(x) \), this means that \( y \) is a function of \( x \), i.e. each input in the domain is mapped into one and only one output \( y \). It is therefore incorrect to use function notation for equations like the one below:

\[
f(x) = x \pm \sqrt{x^2 + 4} \quad \text{WRONG!}
\]

In the equation \( y = x \pm \sqrt{x^2 + 4} \), \( y \) is not a function of \( x \) because of the \( \pm \) sign. It is not always necessary to name the function – we can write \( y = y(x) \). This does not simply mean that \( y \) depends on \( x \), but that \( y \) is a function of \( x \). Therefore, notation like \( y = y(x) \) is also reserved for functions.

Example 4. In each equation, determine whether function notation is used correctly or not.

(a) \( f(x) = x^4 - 3x^2 + 9 \)
(b) \( h(t) = \sqrt{t^2 + 1} \)
(c) \( p(x) = 21 \pm \sqrt{2x + 4} \)
(d) \( g(s) = |2s - 1| \)
(e) \( F(5) = 1, 3 \)

Solution. Functions cannot have outputs described in (c) and (e). Therefore, function notation cannot be used there.

Some specific functions have names that consist of more than one letter. For instance, trigonometric functions are named \( \sin, \cos, \tan, \sec, \csc, \) and \( \cot \) (these are abbreviations of sine, cosine, tangent, secant, cosecant, and cotangent respectively). Among logarithmic functions we have \( \log, \log_2, \ln, \) etc. More often than not, parentheses are not used after these functions if
it is clear what the input is. Thus, \( \sin x \) means \( \sin(x) \), \( \ln x \) means \( \ln(x) \), etc. It is not wrong to write parentheses in those cases, but this is not necessary. Parentheses may be omitted even when the input is not just the independent variable but a term containing the independent variable. Thus, \( \cos 2x \) is usually written for \( \cos(2x) \), which means that the cosine function takes \( 2x \) for its input. Parentheses are needed when the input is an expression consisting of several terms, like in \( \log(2x - 1) \), for instance, where the input is \( 2x - 1 \).

**Example 5.** What is the input in each trigonometric or logarithmic function?

(a) \( \tan \frac{\pi x}{3} \)
(b) \( \log x + 2 \)
(c) \( \log(x + 2) \)
(d) \( \sec 3x \)
(e) \( \csc x - x^2 \)
(f) \( \ln(3x^2 + 1) + x \)

**Solution.** From (a) to (f), the inputs are \( \frac{\pi x}{3} \), \( x \), \( x + 2 \), \( 3x \), \( x \), and \( 3x^2 + 1 \).

**What’s in a name?** The name of a function is not a number. \( f \) is just how a function is called. For instance, \( f(x) \), \( f(0) \), \( f(5) \), but not \( f \), are numbers (provided \( x \), 0 and, 5 are in the domain). A function needs an input to produce a real number as an output. Operations that we can do with numbers to get some number as the result, we cannot do with function names. Thus, \( 3 \cdot g \) is not a number, but \( 3 \cdot g(x) \) is (if \( x \) is in the domain of function \( g \)). In particular, trigonometric and logarithmic functions cannot occur in expressions without an input. When this is not realized, mistakes like the following ones can happen:

\[
\frac{\sin x}{x} = \frac{\sin f}{f} = \sin \text{ WRONG!}
\]

\[
\ln(x + 1) = \ln \cdot x + \ln \cdot 1 \text{ WRONG!}
\]
In the last equation, ln is mistaken for a number which is distributed over $x$ and 1.

**Example 6.** Determine whether each given expression is written correctly or not.

(a) $\frac{x \cos}{x^2 + 3}$

(b) $x \log x + 2$

(c) $x(\log +2)$

(d) $\sec(3x - \pi)$

(e) $\sec(3x) - \pi$

(f) $\sec +3x - \pi$

**Solution.** The expressions (a), (c), and (f) are meaningless. In each of them, the used trigonometric or logarithmic function has no input. □

**EXERCISES 5.2**

1. Identify the name of each function and its input and output.

(a) $G(x) = u$

(b) $y = 7x^2 - 3x + 5$

(c) $y = s(x)$

(d) $f(2) = \sqrt{5}$

(e) $k(x) = 2\sqrt{x} - 19$

2. Write each of the following statements using function notation.

(a) the function is $g$ and its input and output are $t$ and $s$ respectively

(b) the function is $F$ and its input and output are $x$ and $y$ respectively

(c) the function is $p$ and its input and output are $u$ and $3u - 4$ respectively

(d) the function is $h$ and its input and output are $-5$ and $-11$ respectively
3. The function \( f(x) = 3x - 4 \) is to be evaluated (i.e. its output should be found) for \( x = -1, 1, 2 \). Which of the following answers use correct notation?

(a) \( f(-1) = -7, f(1) = -1, f(2) = 2 \)
(b) \( f(x) = -7, f(x) = -1, f(x) = 2 \)
(c) \( -1 = -7, 1 = -1, 2 = 2 \)
(d) \( y = -7, y = -1, y = 2 \)

4. In each equation, determine whether function notation is used correctly or not.

(a) \( g(x) = x \pm \sqrt{2x} \)
(b) \( F(x) = x + |x| \)
(c) \( s(-3) = \pm 8 \)
(d) \( f(t) = \sqrt{t^3 + 1} \)
(e) \( H(0) = \sqrt{4} \)

5. What is the input in each trigonometric or logarithmic function?

(a) \( \ln(2 - 5x) \)
(b) \( \ln 2 - 5x \)
(c) \( \cot(x^2 + x) \)
(d) \( \cot x^2 + x \)
(e) \( \sin \left( x + \frac{\pi}{4} \right) \)
(f) \( \log(1 + 3x) + x^2 \)

6. Determine whether each given expression is written correctly or not.

(a) \( \sqrt{1 + \sin^2} \)
(b) \( (\ln x) \cdot \frac{x}{x + 1} \)
(c) \( \ln \left( \frac{x}{x + 1} \right) \)
(d) \( \ln \cdot \frac{x}{x + 1} \)
(e) \( \cos(2x + \pi) \)
(f) \( \tan +2x + \pi \)
5.3 Types of Functions

Algebraic functions consist of polynomial, rational, and irrational functions. These groups of functions are defined analogously to the corresponding expressions (and equations). Given a function \( f(x) = \text{expression} \), the type of function is the same as the type of the expression on the right-hand side of this equation.

**Example 1.** Identify each function as a polynomial, rational, or irrational function.

\[
\begin{align*}
  f(x) &= 1 + \sqrt{2x^2 + 3} \\
  g(x) &= \sqrt{5} \cdot x + 7 \\
  h(x) &= \frac{4}{3}x^3 - x^2 + 4x^2 + 6x - 1 \\
  k(x) &= \frac{2x - 9}{3x^2 + 5x - 2} \\
  F(x) &= \frac{\sqrt{x} + 4}{x - 2}
\end{align*}
\]

**Solution.** Polynomial functions are \( g \) and \( h \), irrational functions are \( f \) and \( F \), and \( k \) is a rational function.

**Example 2.** Among the given functions, identify those that are constant, linear, or quadratic.

\[
\begin{align*}
  f(x) &= x^4 - 3x^2 + 6 \\
  g(x) &= 1 - \sqrt{5} \\
  h(x) &= \frac{5}{2x - 1} \\
  F(x) &= -x^2 + \frac{2}{3}x - 2 \\
  G(x) &= 6 - 3x \\
  H(x) &= \sqrt{x^2 - 7x + 5} \\
  q(x) &= \sqrt{2} - x^2
\end{align*}
\]

**Solution.** Polynomial functions are \( f \), \( g \), and \( F \), rational functions are \( h \), \( G \), \( H \), and \( q \), and \( k \) is a rational function.
5.3 Types of Functions

Solution. The function \( g \) is constant, \( G \) is linear, and \( F \) and \( q \) are quadratic. □

Transcendental functions form another class of functions. Algebraic functions are those in which the only operations performed on the independent variable are the four basic operations and raising to rational exponents (which include integers). If some other operation is applied to the independent variable, then the function is transcendental. Transcendental functions include exponential, logarithmic, trigonometric, inverse trigonometric, and many other functions.

Example 3. Determine for each function whether it is algebraic or transcendental.

\[
egin{align*}
f(x) &= 2^x - 3x + 5 \\
g(x) &= \sqrt{7} \div (2x - 1) \\
h(x) &= \sin x + 2 \\
F(x) &= x + 2 \log x \\
G(x) &= \sqrt{x^2 - 7x + 5} \\
H(x) &= x^2 + \cos \frac{\pi}{4}
\end{align*}
\]

Solution. Algebraic functions are \( g \), \( G \), and \( H \) (this one in spite of the cosine in it because the cosine is not applied to \( x \); in fact, \( \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \)). The function \( f \) is transcendental because of \( 2^x \), \( h \) because of \( \sin x \), and \( F \) because of \( \log x \). □

Another classification of functions is important in calculus. Given two functions (or expressions) \( h(x) \) and \( g(x) \), the function \( f \) can be classified as follows:

1. \( f \) is a constant multiple of another function if \( f(x) = cg(x) \), where \( c \) is a constant,
2. \( f \) is a product if \( f(x) = g(x) \cdot h(x) \),
3. \( f \) is a quotient if \( f(x) = \frac{g(x)}{h(x)} \),
4. \( f \) is a generalized power if \( f(x) = [g(x)]^n \), where \( n \) is a constant.

This is telling us about the form of the function in terms of the last operation that has to be performed. The list can be extended, but we will not consider other possible forms here. Our goal is not to be able to classify every function but to know to recognize the above four types when they are present. For instance, the function

\[
f(x) = (3x + 5) \sin x
\]

is an example of a product \((3x + 5 \text{ multiplying } \sin x)\). We see a constant multiple of another function in

\[
F(x) = 3(5x - 7)^4,
\]

where the constant is 3 and the other function is

\[
g(x) = (5x - 7)^4,
\]

which happens to be a generalized power. \( F \) is at the same time a product (of a constant function and some other function), but classifying \( F \) as type 1 is more specific. This also shows that the four types are not completely mutually exclusive. Here is another example:

\[
G(x) = \frac{1}{(3x - 4)^2}
\]

can be classified as a quotient, but if we rewrite \( G \) as

\[
G(x) = (3x - 4)^{-2},
\]

we can see that it is also a generalized power. Note the difference between the above \( F \) and

\[
H(x) = [3(5x - 7)]^4,
\]

which is a generalized power.

The order of operations can help distinguish between types 1–4. Consider

\[
f(x) = (3x - 5) \sin x
\]

again. Suppose the problem is to find \( f(1) \). We have to calculate \( 3 \cdot 1 - 5 \) and \( \sin 1 \) separately and then to multiply the two numbers. Therefore,
multiplication is the last operation that should be done and this classifies \( f \) as a product. *The last operation identifies the function type.*

**Example 4.** Classify each function, if possible, as a constant multiple of another functions, product, quotient, or generalized power.

\[
\begin{align*}
f(x) &= 2^x - 3x + 5 \\
g(x) &= \frac{\sqrt{7}}{2x - 1} \\
h(x) &= \sqrt{\sin x + 2} \\
F(x) &= (x + 2)^6(5 - 2 \log x)^3 \\
G(x) &= 5\sqrt{x^2 - 7x + 5} \\
H(x) &= \frac{x^2 - 3x + 8}{x - 4}
\end{align*}
\]

**Solution.** The function \( f \) does not belong to any of the four types. \( g \) is a quotient and so is \( H \). The function \( h \) is a generalized power since \( h(x) = (\sin x + 2)^{1/2} \). \( F \) is a product and \( G \) is a constant multiple of another function.

\( \square \)

**EXERCISES 5.3**

1. Identify each function as a polynomial, rational, or irrational function.

\[
\begin{align*}
f(x) &= 2x^2 + 3x - 7 \\
g(x) &= \sqrt{5} \\
h(x) &= x^6 - \frac{1}{6}x^2 \\
k(x) &= \sqrt{2x - 93x^2 + 5x - 2} \\
F(x) &= x^2 + 3\sqrt{x} - 7
\end{align*}
\]

2. Among the given functions, identify those that are constant, linear, or quadratic.

\[
\begin{align*}
f(x) &= \pi \\
g(x) &= 1 - 3x^2
\end{align*}
\]
3. Determine for each function whether it is algebraic or transcendental.

\[ f(x) = x^4 - 3x + 5 \]
\[ g(x) = x^3 + 1 \]
\[ h(x) = x^2 - x + \sec(x + 2) \]
\[ F(x) = \frac{\sqrt{x}}{2x - 7} \]
\[ G(x) = \ln(x^2 - x) \]
\[ H(x) = x^3 + 2^{10} \]

4. Classify each function, if possible, as a constant multiple of another functions, product, quotient, or generalized power.

\[ f(x) = x(2x - 3x + 5) \]
\[ g(x) = \sqrt{7} \cdot (2x^3 - 1) \]
\[ h(x) = 2 + \sin x \]
\[ F(x) = [(2x + 1)\tan x]^5 \]
\[ G(x) = \frac{(x + 2)(x^2 - 3)}{x^2 - 4x + 5} \]
\[ H(x) = -3(x^2 + 1)^3 \]

5.4 Discussing the Domain

The domain is the set of \( x \)-values for which the function is defined. We have mentioned in Section 5.1 that when a function \( y = f(x) \) is given as an equation, then the domain is usually not provided. It is understood that the domain consists of all \( x \)-values that the operations in \( f(x) \) can be performed on. The following is a list of simpler operations (we shall not discuss trigonometric functions here) that are undefined:
• division by 0,
• even roots of negative numbers,
• logarithms of non-positive numbers.

Finding the domain of a given function is a typical mathematical problem. When solving this problem, we have to keep the above list in mind.

If the given function contains a fraction, the denominator cannot equal 0. Therefore, to set up this part of domain discussion, we have to set each denominator equal to 0, solve those equations, and exclude all solutions from the domain. For instance, if

\[ f(x) = \frac{3x - 1}{x + 2} + \frac{1}{x^2 - 1}, \]

we have to solve the equations \( x + 2 = 0 \) and \( x^2 - 1 = 0 \). The solutions are \( x = -2, -1, 1 \), which give the domain

\[ D = \text{all real numbers } x \neq -2, -1, 1. \]

This verbal description of the domain is perfectly acceptable, but interval notation may be used as well:

\[ D = (-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, \infty). \]

**Example 1.** Which of the following statements is an acceptable answer when discussing the domain of the function \( f(x) = \frac{3x^2 - 6}{x^2 + 3x + 2} \)?

(a) \( D = \mathbb{R} \neq -2, -1 \)
(b) \( D = \text{all real } x \neq -2, -1 \)
(c) \( D = \text{all real } x \text{ except } x \neq -2, -1 \)
(d) \( D = \text{all real } x \text{ except } x = -2, -1 \)
(e) \( D = (-\infty, -2) \cup (-1, \infty) \)
(f) \( D = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty) \)
(g) \( D = \text{all real } x \neq 0 \)

Solution. The denominator of \( f \) is equal to 0 if \( x = -2, -1 \). Therefore, the answers (b), (d), and (f) are acceptable. Answer (a) contains a notational mistake: \( \mathbb{R} \) is the set of all real numbers, which definitely does not equal \(-2, -1\). What is meant is “all reals not equal to \(-2, -1\)”, but \( \mathbb{R} \) does not stand for the phrase “all reals”. Answer (c) contains a logical mistake: “all real \( x \) except \( x \neq -2, -1 \)” leaves the set \( \{-2, -1\} \) for the domain. The answer is not complete in (e) – the function is also defined on the interval \((-2, -1)\). Finally, (g) shows \( x \) mistaken for the denominator. \( \Box \)

Example 2. Set up the equations (if any) that need to be solved in order to find the domain of each function.

\[
\begin{align*}
f(x) &= \frac{3x + 5}{4x - 1} \\
g(x) &= \frac{1}{x^3 + x^2} - \frac{5x}{x - 7} \\
h(x) &= \frac{9x + 4}{8} \\
F(x) &= \frac{x^3 - 8}{x^2 + 1}
\end{align*}
\]

Solution. There is no need for an equation when discussing the domain of \( h \) – its denominator is 8, which is not equal to 0. For the function \( F \), we may set up the equation \( 4x - 1 = 0 \), but we may immediately realize that this denominator is never equal to 0. On the other hand, there are some equations to be solved in order to find the domain of \( f \) and \( g \). The equations are \( 4x - 1 = 0 \) for \( f \) and \( x^3 + x^2 = 0 \) and \( x - 7 = 0 \) for \( g \). \( \Box \)

If the given function contains an even radical, its radicand has to be greater or equal to 0. This means that we should set up the inequality \( \text{radicand} \geq 0 \) and solve it. If the inequality is non-linear, we have to use the sign chart, see Section 4.3. Note that this only applies to even radicals – odd radicals are defined for any sign of the radicand.

Example 3. Set up the inequalities (if any) that need to be solved in order to find the domain of each function.

\[
\begin{align*}
f(x) &= \sqrt{x} + \sqrt{x - 6} \\
g(x) &= \sqrt[3]{x^2 - 2x + 7} - \sqrt{3x^2 - 5}
\end{align*}
\]
5.4 Discussing the Domain

\[ h(x) = 3x^2 - x + 2 + \sqrt{x} \]
\[ F(x) = \sqrt{5x + 1} - \sqrt{x^2 - 3} \]

Solution. The inequality needed for the function \( f \) is \( x \geq 0 \), which is already solved for \( x \). The function \( g \) requires \( 3x^2 - 5 \geq 0 \). There is no even radical in \( h \), so no inequality needs to be set up. In \( F \), there are two inequalities that need to be solved simultaneously: \( 5x + 1 \geq 0 \) and \( x^2 - 3 \geq 0 \).

Consider the function \( f(x) = \sqrt{2x - 4} \). A typical misconception when discussing the domain of this function is to set up the inequality \( \sqrt{2x - 4} \geq 0 \). The above statement is not wrong (actually, it is always true since even radicals cannot produce negative numbers), but this is not what we are supposed to discuss. It is the \textit{radicand} that matters here, therefore, \( 2x - 4 \geq 0 \) is the correct inequality.

If the given function contains a logarithm, the expression that the logarithm is applied to has to be positive. To discuss the domain in this case, we again have to set up an inequality and solve it. The inequality is of the form \( \text{expression} > 0 \), where the \textit{expression} is the input of the logarithmic function.

Example 4. Set up the inequalities (if any) that need to be solved in order to find the domain of each function.

\[ f(x) = x + \log(x - 6) \]
\[ g(x) = 3x^2 - x + \ln 5 \]
\[ h(x) = (x + \ln x)^4 \]
\[ F(x) = \log(x^2 - 1) + \log(x^2 + 1) \]

Solution. The inequality needed for the function \( f \) is \( x - 6 > 0 \). The function \( g \) is immediately defined for all real numbers – it is just a polynomial because \( \ln 5 \) is a constant. The function \( h \) requires \( x > 0 \). As for \( F \), we may set up two inequalities, \( x^2 - 1 > 0 \) and \( x^2 + 1 > 0 \), but the latter is always true. Therefore, only \( x^2 - 1 > 0 \) has to be solved and this should be done using
the sign chart.

If the given function contains any combination of the three operations discussed above, all the necessary requirements have to be satisfied at the same time. A drastic situation is when the operations are combined so that the function is not defined for any real number, that is, \( D = \emptyset \). In such a case, the function does not exist. \( f(x) = \ln x + \ln(-x) \) illustrates this – on the one hand \( x > 0 \) has to hold true, and on the other, \( -x > 0 \), i.e. \( x < 0 \). It is impossible that \( x > 0 \) and \( x < 0 \) at the same time. We can put whatever we want to on the paper, but the question is whether this makes sense or not. This “function” does not.

If the function does not contain any of the three operations which may be undefined, then we can expect it to be defined for all real numbers, \( D = \mathbb{R} \). For instance, polynomials are such functions. Or, the domain of 

\[
f(x) = 2x^3 - 4x^2 + \frac{6}{5}x - 1 + \sqrt[3]{x^2 - 3x} + 8
\]

is immediately \( D = \mathbb{R} \) since this function does not involve any of the three “problematic” operations. However, it is important not to jump into conclusions here and to inspect the function carefully. In the following example, after seeing that 

\[
g(x) = \sqrt[3]{\frac{x + 1}{x}}
\]

is a cube root, we may think that \( D = \mathbb{R} \) because “cube roots are always defined”. It is true that taking the cube root is an operation that is always defined, but what if the radicand is not defined? The radicand of the above function \( g \) is a fraction and it is undefined when \( x = 0 \). Therefore, the domain of \( g \) is all real numbers \( x \neq 0 \). Similarly, when we say that exponential functions are always defined, we mean that it is always possible to do the operation of raising a positive constant base to any real number. But, if the exponent is undefined, then it is not a real number and then the exponential function is not defined. The domain of the function 

\[
h(x) = 2\sqrt{x}
\]

is not all reals, not because of the operation of raising 2 to a real number, but because the square root in the exponent requires that \( x \geq 0 \).

Let us now discuss some examples of functions with radicals in the denomi-
nator. Consider
\[ f(x) = \frac{1}{\sqrt{2x-8}} \quad \text{and} \quad g(x) = \frac{1}{\sqrt{2x-8}}. \]

It would again be wrong to say that \( f \) is defined for all reals because of the cube root. The cube root is in the denominator and, although it is always defined, the fraction is not when \( x = 4 \). The domain of \( f \) is all real numbers \( x \neq 4 \). Similarly, in \( g \) we have a combination of the square root and a fraction. Because of the square root, \( 2x - 8 \geq 0 \) has to be satisfied, but the fraction requires that \( 2x - 8 \neq 0 \). Combining these two requirements, we get \( 2x - 8 > 0 \), that is, \( x > 4 \), which describes the domain of \( g \).

**Example 5.** Set up all inequalities (if any) that need to be solved in order to find the domain of each function.

\[
\begin{align*}
  f(x) &= \log \left(4 - \sqrt{x + 5}\right) \\
  g(x) &= \frac{x^2 - 2x + 1}{\sqrt{x^2 - 4}} \\
  h(x) &= 2x + 9 - \sqrt{x^2 + 6x - 1} + 3^{1/(x-1)} \\
  F(x) &= \frac{\ln x}{\sqrt{5x + 1}} \\
  G(x) &= \sqrt[3]{\frac{7x - 3}{x^2 - 5x + 6}} \\
  H(x) &= \frac{\sqrt{7x - 3}}{x^2 - 5x + 6}
\end{align*}
\]

**Solution.** The function \( f \) requires \( x - 5 \geq 0 \) and \( 4 - \sqrt{x + 5} > 0 \). For \( g \), we need \( x^2 - 4 \neq 0 \). We should, therefore, solve the equation \( x^2 - 4 = 0 \) and exclude the solutions from the domain. The inequalities for \( h \) are \( x^2 + 6x - 1 \geq 0 \) and \( x - 1 \neq 0 \). The function \( F \) requires that \( x > 0 \) and \( 5x + 1 > 0 \). For \( G \), we need \( x^2 - 5x + 6 \neq 0 \). Finally, \( H \) requires \( 7x - 3 \geq 0 \) and \( x^2 - 5x + 6 \neq 0 \).

**EXERCISES 5.4**

1. Which of the following statements is an acceptable answer when discussing the domain of the function \( f(x) = \frac{3x - 6}{x^2 - x - 6} \)?
(a) \( D = \text{all real } x \text{ except } x \neq -2, -1 \)
(b) \( D = (-\infty, -2) \cup (-2, 3) \cup (3, \infty) \)
(c) \( D = \mathbb{R} \neq -2, 3 \)
(d) \( D = \text{all real } x \text{ except } x = -2, 3 \)
(e) \( D = (-\infty, -2) \cup (3, \infty) \)
(f) \( D = \text{all real } x \neq -2, 3 \)
(g) \( D = \text{all real } x \neq 2, -2, 3 \)

2. Set up the equations (if any) that need to be solved in order to find the domain of each function.

\[
f(x) = \frac{3x - 5}{4x^2 + 1} \\
g(x) = \frac{6x^2 + 3}{x^2 - x - 12} \\
h(x) = \frac{x^3 - x^2}{x^3 - 8} \\
F(x) = \frac{5}{x^2 - 1} + \frac{3x}{4x - 3}
\]

3. Set up the inequalities (if any) that need to be solved in order to find the domain of each function.

\[
f(x) = \sqrt{x + 4} + \sqrt{x^2 + x} \\
g(x) = \sqrt{x^2 - 2x + 7} - \sqrt{3x^2 - 5} \\
h(x) = \sqrt{5 + 3\sqrt{7x^2 - x + 6}} \\
F(x) = x^2 + 5x\sqrt{x} - 3x + 1
\]

4. Set up the inequalities (if any) that need to be solved in order to find the domain of each function.

\[
f(x) = 3x + 7 - \log 6 \\
g(x) = \ln(2x^2 + 5) \\
h(x) = x \log(3x^2 - 4) \\
F(x) = \ln(x + 1) - \ln(x^2 + 5x + 6)
\]

5. Set up all inequalities (if any) that need to be solved in order to find the domain of each function.

5.4 Discussing the Domain

\[
f(x) = \frac{4 - \sqrt{x} + 5}{x^3 - x^2}
\]

\[
g(x) = \sqrt{\frac{x^2 - 2x + 1}{3x - 5}}
\]

\[
h(x) = \ln\left(2x + 9 - \frac{1}{x}\right)
\]

\[
F(x) = 2 \sqrt[3]{x^2 - 9}
\]

\[
G(x) = \frac{8x - 1}{\sqrt{x^2} + 2x + 1}
\]

\[
H(x) = \frac{\sqrt[5]{7x - 3}}{1 + \log x}
\]