The Cobb-Douglas Production Function

\[ Y = AK^\alpha L^{1-\alpha} \]

**History**

- Developed by Paul Douglas and C. W. Cobb in the 1930’s
  - Douglas went on to be professor at Chicago and U.S. Senator
  - Cobb - ??

**The General Problem**

- An increase in a nation’s capital stock or labor force means more output.
- Is there a mathematical formula that relates capital, labor and output?

**The General Form**

\[ Y_t = A_t K_t^\alpha L_t^{1-\alpha} \]

**Increasing Capital**

\[ Y_o = AK_o^\alpha L_o^{1-\alpha} \]

\[ Y = A(2K_o)^\alpha L_o^{1-\alpha} \]
The Cobb-Douglas Production Function

Increasing Capital

\[ Y_o = AK_o^\alpha L_o^{1-\alpha} \]

\[ Y = A(2K_o)^\alpha L_o^{1-\alpha} = 2^\alpha AK_o^\alpha L_o^{1-\alpha} = 2^\alpha Y_o \]

Diminishing returns to proportion

Increasing Labor

\[ Y_o = AK_o^\alpha L_o^{1-\alpha} \]

\[ Y = AK_o^\alpha (2L_o)^{1-\alpha} = 2^{1-\alpha} AK_o^\alpha L_o^{1-\alpha} = 2^{1-\alpha} Y_o \]

Diminishing returns to proportion

Increasing Both

\[ Y_o = AK_o^\alpha L_o^{1-\alpha} \]

\[ Y = A(2K_o)^\alpha (2L_o)^{1-\alpha} \]
The Cobb-Douglas Production Function

**Increasing Both**

\[ Y_o = AK_o^\alpha L_o^{1-\alpha} \]

\[ Y = A(2K_o)^\alpha (2L_o)^{1-\alpha} = 2 \ AK_o^\alpha L_o^{1-\alpha} = 2 \ Y_o \]

**Constant returns to scale**

**Substitution**

\[ Y_o = AK_o^\alpha L_o^{1-\alpha} \]

\[ Y = A(2K_o)^\alpha (xL_o)^{1-\alpha} = Y_o \]

**Capital and Labor Can be Substituted**

**An Illustration**

\[ Y_t = A_t K_t^{1/2} L_t^{1/2} \]

**An Illustration**

\[ Y = AK^{1/2} L^{1/2} \]

\[ Y = A \sqrt{KL} \]
An Illustration

\[ Y = A\sqrt{KL} \]

\[ A = 3 \quad L = 10 \quad K = 10 \]

Y = 3 \sqrt{(10)(10)} = 3 \sqrt{100} = 30

Doubling Capital

\[ Y = 3\sqrt{(20)(10)} = 3\sqrt{200} = 30\sqrt{2} \approx 42 \]

Constant Returns to Scale

\[ Y = 3\sqrt{(20)(20)} = 3\sqrt{400} = 60 \]

Substitution

\[ Y = 3\sqrt{(20)(x)} = 30 \]

\[ x = 5 \]

Estimation

\[ Y_i = A_i K_i^\alpha L_i^{1-\alpha} \]

\[ \log(Y_i) = \log(A_i) + \alpha \log(K_i) + (1 - \alpha) \log(L_i) \]
Estimation
\[ \log(Y_t) = C + t + \beta_1 \log(K_t) + \beta_2 \log(L_t) + \varepsilon_t \]

Statistical issues abound!

Best Estimate
\[ \alpha \approx \frac{1}{3} \]

Factor Payments
- \(\alpha = \) % of Income going to owners of capital
- \(1-\alpha = \) % of Income going to workers

How well does it work?
\[ Y_t = A_t K_t^\alpha L_t^\beta \]

You can’t beat something with nothing

CES Production Function
\[ \sigma = \frac{\% \Delta \left( \frac{K}{L} \right)}{\% \Delta \left( \frac{W}{r} \right)} \]
Suppose a 10% increase in wage rate leads to a 10% increase in capital labor ratio. \( \sigma = 1. \)

Suppose a 10% increase in wage rate leads to a 5% increase in capital labor ratio. \( \sigma = \frac{1}{2}. \)

In the Cobb-Douglas Production Function, \( \sigma = 1. \)

The CES allows for a different elasticity of substitution. Little gained.

\[ \% \Delta \left( \frac{W}{r} \right) \]

\[ \% \Delta \left( \frac{W}{r} \right) \]

\[ K = aY \]

\[ L = bY \]
The Cobb-Douglas Production Function

Leontief Production Function

• $K = aY$
• $L = bY$

The Cobb-Douglas Production Function

Leontief Production Function

• $K = aY$
• $L = bY$

$\sigma = 0$

Doesn’t work. We can and do substitute labor for capital all the time

Other Factors?

$Y_t = A_t K_t^{\alpha} L_t^\beta LND_t^{1-\alpha-\beta}$

And in Conclusion…

$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$

End

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