Introduction to Analysis of Variance (ANOVA)

The Structural Model, The Summary Table, and the One-Way ANOVA

Departing the $t$-Distribution

- Although the $t$-Test is commonly used, it has limitations
  - Can only test differences between 2 groups
    - High school class?? College year??
  - Can examine ONLY the effects of 1 IV on 1 DV
  - Limited to single OR repeated measures designs
Limitations of the $t$-Test

- Testing differences between group means
  - IV: Gender (Male & Female)
  - IV: High-school class (First-year, Sophomore, Junior, & Senior)

  - Using the $t$-Test, we must either “collapse” categories… or not run the analysis!

Limitations of the $t$-Test

- 1 Independent Variable
  - Gender differences in depression
    - IV: Gender (Male & Female)
    - DV: Level of depression (BDI score)

- 2 Independent Variables
  - Gender and social support on depression
    - IV: Gender (Male & Female)
    - IV: Social support (High, Medium, & Low)
    - DV: Level of depression (BDI score)
Limitations of the *t*-Test

• 2 or more Independent Variables
  – Simultaneously examine the impact of 2 or more IVs on a single DV
  – Examine how the effects of 2 or more IVs COMBINE to affect a single DV

Limitations of the *t*-Test

• Single time point OR repeated measures designs
  – 1 group at 2 time points = repeated measures
  – 2 groups at 1 time point = independent groups

• Single time point AND repeated measures designs
  – 2 or more groups at 2 or more time points
The Analysis of Variance (ANOVA)

- The ANOVA test can examine data that the $t$-Test cannot
- Probably the most commonly abused statistical test
- Many varieties of ANOVA
  - One-Way
  - Factorial
  - Repeated Measures
  - Mixed-Model

Varieties of ANOVA

- One-Way ANOVA
  - 1 continuous Dependent Variable
  - 1 Independent Variable consisting of 2 or more “categorical” groups
    - The one-way ANOVA with 2 groups is equivalent to the independent groups $t$-Test
Varieties of ANOVA

• Factorial ANOVA
  – 1 continuous Dependent Variable
  – 2 or more Independent Variables consisting of 2 or more “categorical” groups for each IV
    • 2 IVs = Two-Way Factorial ANOVA
    • 3 IVs = Three-Way Factorial ANOVA

  – We call these “factorial” designs because EACH level of each IV is paired with EVERY level of ALL other IVs

2 x 2 Contingency Table

<table>
<thead>
<tr>
<th></th>
<th>IV 1</th>
<th></th>
<th>IV 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Level 1</td>
<td></td>
<td>Level 1</td>
</tr>
<tr>
<td>DATA</td>
<td>DATA</td>
<td></td>
<td>DATA</td>
</tr>
<tr>
<td></td>
<td>Level 2</td>
<td></td>
<td>Level 2</td>
</tr>
<tr>
<td>DATA</td>
<td>DATA</td>
<td></td>
<td>DATA</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Row 1Total</td>
<td></td>
</tr>
<tr>
<td>DATA</td>
<td></td>
<td>Row 2Total</td>
<td></td>
</tr>
<tr>
<td>Col 1Total</td>
<td></td>
<td>Col 2Total</td>
<td></td>
</tr>
</tbody>
</table>

Note: Each Level 1 of IV 1 is paired with BOTH Level 1 and Level 2 of IV 2
2 x 2 Contingency Table

<table>
<thead>
<tr>
<th>Gender</th>
<th>Social Support</th>
<th>Row 1 Total</th>
<th>Row 2 Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>High</td>
<td>DATA</td>
<td>DATA</td>
</tr>
<tr>
<td></td>
<td>Low</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>High</td>
<td>DATA</td>
<td>DATA</td>
</tr>
<tr>
<td></td>
<td>Low</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Col 1 Total</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Col 2 Total</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Each Level 1 of IV 1 is paired with BOTH Level 1 and Level 2 of IV 2

Varieties of ANOVA

- Repeated Measures ANOVA
  - 1 continuous Dependent Variable
  - 1 Independent Variable consisting of 2 or more “categorical” time points
    - The DV is assessed at EACH time point
Varieties of ANOVA

- Mixed-Model ANOVA
  - 1 continuous Dependent Variable
  - 1 or more Independent Variables consisting of 2 or more “categorical” groups
  - 1 Independent Variable consisting of 2 or more “categorical” time points
    • The DV is assessed at EACH time point

ANOVA

- For this class, we’re going to stick to fairly simple ANOVA models
  - One-Way ANOVA
  - Factorial ANOVA (I hope…)
Choosing the Best Test

The Underlying Model

- A statistical model by example:
  - Assume: the average 18 year old human being weight approximately 138 pounds
    - Men, on average, weigh 12 pounds more than the average human weight
    - Women, on average, weigh 10 pounds less than the average human weight
The Underlying Model

• For any given human being, I can break weight down into 3 components:
  – Average weight for all individual
    • 138 lbs
  – Average weight for each group
    • Men: +12 lbs
    • Women: - 10 lbs
  – The individual’s unique difference

• Male weight
  Weight = 138 lbs + 12 lbs + uniqueness

• Female weight
  Weight = 138 lbs – 10 lbs + uniqueness

• If you understand this process, you understand the basic theory behind the ANOVA
Partitioning Variance

- The idea behind the ANOVA test is to divide or separate (partition) variance observed in the data into categories of what we CAN and what we CANNOT explain.

The Structural Model

- Mathematically, we partition the total variance of our data using the structural form of the ANOVA model:
  \[ X_{ij} = \mu + \tau_j + \varepsilon_{ij} \]
  - The structural model translates as follows: The score for any single individual is equal to the sum of the population mean plus the mean of the group plus the individual’s unique contribution.
The Structural Model

• For our weight example:
  – \( \mu \) = population weight = 138 lbs
  – \( \tau \) = group difference in weight = 12 or 10 lbs
  – \( \varepsilon \) = “unique” contribution of an individual’s score

  – \( \mu \) & \( \tau \) can be explained
  – \( \varepsilon \) cannot be explained…

Uniqueness

• Oftentimes, we value our uniqueness…
  – In statistics, unique variance is BAD
  – Since we can’t explain unique variance, we call it “error”
  – Thus, the ANOVA seeks to examine the relative proportion of explainable variance in our data to the unexplainable variance
Assumptions of the ANOVA

• Owing to the mathematical construction of the ANOVA, the underlying assumptions of the test are very important
  – Homogeneity of variance
  – Normality
  – Independence of Observations
  – The Null Hypothesis

Homogeneity of Variance

• Homogeneity of variance refers to the variance for each group being equal to the variance of every other group
  – Really, we mean that the variance of each group is equal to the variance of the error for the total analysis
  – $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_e^2$
Homogeneity of Variance

- Heterogeneity of variance is another BAD thing
  - Heterogeneous variances can greatly influence the results you obtain, making it either more or less likely that you will reject $H_0$
  - Tests of homogeneity of variance
  - Visual inspection of variances

Normality

- The ANOVA procedure assumes that scores are normally distributed
  - More accurately, it assumes that ERRORS are normally distributed
  - Random sampling and random assignment
  - Lacking normality, consider mathematical transformations
    - Logarithmic & square root transformations
Independence of Observations

• Simple: The scores for 1 group are not dependent on the scores from another group
  – Don’t share subjects between groups
  – If violated… why?

The Null Hypothesis

• Less an assumption and more a theoretical point:
  – $H_0$: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$
• This is almost ALWAYS the basic form of your null hypothesis…
Calculating the One-Way ANOVA

- In order to calculate the One-Way ANOVA statistic, we need to complete a number of intermediate steps.
- Because there are several intermediate steps, we keep track of our progress with something called a summary table.

The Summary Table

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Sum of Squares (SS)</th>
<th>Mean Square (MS)</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
One-Way ANOVA: Partitioning Variance

- The idea behind the ANOVA test is to divide or separate (partition) variance observed in the data into categories of what we CAN and what we CANNOT explain.

![Diagram showing total variance partitioned into treatment and error categories.]

**The Summary Table**

<table>
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<th>Mean Square (MS)</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>(k-1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>k(n-1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>(N-1)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: \( df_{\text{Treatment}} + df_{\text{Error}} = df_{\text{Total}} \)
The Summary Table

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<th>F</th>
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</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>(k-1)</td>
<td>( SS_{\text{Treatment}} = \sum n(X_{ij} - \bar{X}_j)^2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>(k(n-1))</td>
<td>( SS_{\text{Error}} = SS_{\text{Total}} - SS_{\text{Treatment}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>(N-1)</td>
<td>( SS_{\text{Total}} = \sum (X_{ij} - \bar{X}<em>.)^2 ) ( SS</em>{\text{Total}} = \sum x^2 - \frac{(\sum x)^2}{N} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Sum of Squares

- Note:
  - \( \bar{X}_j \) = The treatment group mean
  - \( \bar{X}_. \) = The grand mean (mean of all scores)
  - \( X_{ij} \) = Each individual score
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<tr>
<td>Treatment</td>
<td>(k-1)</td>
<td>[ SS_{\text{Treatment}} = \sum n(X_j - \bar{X})^2 ]</td>
<td>[ MS_{\text{Treatment}} = \frac{SS_{\text{Treatment}}}{df_{\text{Treatment}}} ]</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>k(n-1)</td>
<td>[ SS_{\text{Error}} = SS_{\text{Total}} - SS_{\text{Treatment}} ]</td>
<td>[ MS_{\text{Error}} = \frac{SS_{\text{Error}}}{df_{\text{Error}}} ]</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>(N-1)</td>
<td>[ SS_{\text{Total}} = \sum (X_{ij} - \bar{X}_{..})^2 ]</td>
<td>[ SS_{\text{Total}} = \sum x^2 - \frac{(\sum x)^2}{N} ]</td>
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</tr>
<tr>
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<td>[ MS_{\text{Error}} = \frac{SS_{\text{Error}}}{df_{\text{Error}}} ]</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>(N-1)</td>
<td>[ SS_{\text{Total}} = \sum (X_{ij} - \bar{X}_{..})^2 ]</td>
<td>[ SS_{\text{Total}} = \sum x^2 - \frac{(\sum x)^2}{N} ]</td>
<td></td>
</tr>
</tbody>
</table>
The Equations

\[
SS_{Treatment} = \sum n(\bar{X}_j - \bar{X}_{..})^2
\]

\[
SS_{Treatment} = \sum [n(\bar{X}_j - \bar{X}_{..})^2 + n(\bar{X}_j - \bar{X}_{..})^2 + ...]
\]

\[
SS_{Total} = \sum x^2 - \left(\frac{\sum x}{N}\right)^2
\]

\[
MS_{Treatment} = \frac{SS_{Treatment}}{df_{Treatment}}
\]

\[
SS_{Error} = SS_{Total} - SS_{Treatment}
\]

\[
MS_{Error} = \frac{SS_{Error}}{df_{Error}}
\]

\[
F = \frac{MS_{Treatment}}{MS_{Error}}
\]

Examples

• Anorexia Nervosa 3 Group Tx Example
• Anorexia Nervosa 2 Group Tx Example – Compared to Independent Groups t-Test
• GRE-V Test Preparation Example