Predicates and Quantifiers

Section 1.4

Section Summary

- Predicates
- Propositional functions
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan’s Laws for Quantifiers
- Translating English to Logic

Propositional Logic Not Enough

- If we have:

  "All men are mortal."

  "Socrates is a man."

- Does it follow that "Socrates is mortal?"
- Can't represent this in propositional logic. Need a language that talks about objects, their properties, and their relations.
- Later we'll see how to draw inferences.

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: $P(x), M(x), R(x,y)$ statements that are either true or false based on the value of its variables
  - Quantifiers (*to be covered in a few slides*):
- *Propositional functions* are a generalization of propositions.
  - They contain variables and a predicate, e.g., $P(x)$
  - They become propositions (and have truth values) when
    - their variables are replaced by a value from their *domain*, or
    - their variables are *bound* by a *quantifier*

Propositional Functions

The statement $P(x)$ is said to be the value of the propositional function P at x.

Ex: Let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

- P(-3) is false.
- P(0) is false.
- P(3) is true.

Often the domain is denoted by U. So in this example U is the integers.

Examples of Propositional Functions

Ex: Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find the truth value of:

- R(2,-1,5)

  Solution: F

- R(3,4,7)

  Solution: T

- R(x, 3, z)

  Solution: Not a Proposition

Examples of Propositional Functions

Ex: Let $Q(x, y, z)$ denote "$x - y = z$", with U as the integers. Find the truth value of:

- Q(2,-1,3)

  Solution: T

- Q(3,4,7)

  Solution: F

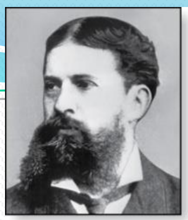
- Q(x, 3, z)

  Solution: Not a Proposition

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- **Ex**: If *P(x)* denotes "$x > 0$," find these truth values:
  - $P(3) \vee P(-1)$ T
  - $P(3) \wedge P(-1)$ F
  - $P(3) \rightarrow P(-1)$ F
  - $P(-1) \rightarrow P(3)$ T
- Expressions with variables are not propositions and therefore do not have truth values. For example,
  - $P(3) \wedge P(y)$
  - $P(x) \rightarrow P(y)$
- When used with quantifiers, these expressions (propositional functions) become propositions.

Quantifiers



Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including "*all*" and "*some*":
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier*, "For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for every (all) x in the *domain*.
- $\exists x\, P(x)$ asserts $P(x)$ is true for some x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

$\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

- If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
- If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
- If $P(x)$ denotes "x is even" and U is the integers, then $\forall x P(x)$ is false.

Existential Quantifier

$\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

- If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x P(x)$ is true. It is also true if U is the positive integers.
- If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x P(x)$ is false.
- If $P(x)$ denotes "x is even" and U is the integers, then $\exists x P(x)$ is true.

Uniqueness Quantifier (*optional*)

- $\exists! x\, P(x)$ means that $P(x)$ is true for <u>one and only one</u> x in the domain.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique x such that $P(x)$."
  - "There is one and only one x such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then $\exists! x\, P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then $\exists! x\, P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique x such that $P(x)$ can be expressed as:

$$\exists x\, (P(x) \land \forall y\, (P(y) \rightarrow y = x))$$

Thinking about Quantifiers

- When the domain is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step P(x) is true, then $\forall x\, P(x)$ is true.
  - If at a step P(x) is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, P(x) is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which P(x) is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depends on both the propositional function $P(x)$ and on the domain U.

Examples:

- If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
- If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
- If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
- If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \lor Q(x)$ means $(\forall x\, P(x)) \lor Q(x)$
- $\forall x\, (P(x) \lor Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \lor Q(x)$ when they mean $\forall x\, (P(x) \lor Q(x))$.

Translating from English to Logic

Example: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution: First decide on the domain *U*.

- **Solution 1**: If *U* is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.
- **Solution 2**: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution: First decide on the domain U.

- **Solution 1**: If U is all students in this class, translate as $\exists x\, J(x)$
- **Solution 2**: But if U is all people, then translate as $\exists x\, (S(x) \wedge J(x))$

  $\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Logical Equivalences

- Assume S and T are two statements involving predicates and quantifiers.
- S and T are *logically equivalent* if and only if they have the same truth value for every predicate substituted into these statements and for every domain used, denoted $S \equiv T$.
- **Ex**: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite
  - a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers for each element in the domain
  - an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers for each element in the domain.

- **Ex**: If *U* consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$
$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and the domain is students in your class.

- Negating the original statement gives:
  - "It is not the case that every student in your class has taken Java."
  - This implies that "There is a student in your class who has not taken Java."

Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions (*continued*)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives
  - "It is not the case that there is a student in this class who has taken Java."
  - This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

The rules for negating quantifiers are:

	When true?	**When false?**
$\neg\exists x P(x) \equiv \forall x \neg P(x)$	For every x, P(x) is false.	There is an x for which P(x) is true.
$\neg\forall x P(x) \equiv \exists x \neg P(x)$	There is an x for which P(x) is false.	P(x) is true for every x.

Examples Translating from English to Logic

- "Some student in this class has visited Mexico."

Solution: Let U be all people.

$M(x)$ = "x has visited Mexico"

$S(x)$ = "x is a student in this class,"

$$\exists x \ (S(x) \land M(x))$$

- "Every student in this class has visited Canada or Mexico."

Solution: Add $C(x)$ = "x has visited Canada."

$$\forall x \ (S(x) \rightarrow (M(x) \lor C(x)))$$

Additional Examples

Translate these statements into logic, where the domain consists of all animals and R(x)= “x is a rabbit” and H(x)=“x hops”.

1. Every animal is a rabbit and hops.
2. There exists an animal such that if it is a rabbit then it hops.
3. Every rabbit hops.
4. Some hopping animals are rabbits.
5. There exists an animal that is a rabbit and hops.
6. Some rabbits hop.
7. If an animal is a rabbit, then that animal hops.
8. All rabbits hop.

Translate these statements into logic, where the domain consists of all animals and R(x)= "x is a rabbit" and H(x)="x hops".

1. Every animal is a rabbit and hops. $\forall x(R(x) \land H(x))$
2. There exists an animal such that if it is a rabbit then it hops. $\exists x(R(x) \rightarrow H(x))$
3. Every rabbit hops. $\forall x(R(x) \rightarrow H(x))$
4. Some hopping animals are rabbits. $\exists x(R(x) \land H(x))$
5. There exists an animal that is a rabbit and hops. $\exists x(R(x) \land H(x))$
6. Some rabbits hop. $\exists x(R(x) \land H(x))$
7. If an animal is a rabbit, then that animal hops. $\forall x(R(x) \rightarrow H(x))$
8. All rabbits hop. $\forall x(R(x) \rightarrow H(x))$

Let Q(x) be the statement "$x \geq 2x$" and the domain consist of all integers. What are these truth values?

1. Q(0)
2. Q(-1)
3. Q(1)
4. $\forall x Q(x)$
5. $\exists x Q(x)$
6. $\exists x \neg Q(x)$
7. $\forall x \neg Q(x)$

Let $Q(x)$ be the statement "$x \geq 2x$" and the domain consist of all integers. What are these truth values?

1. $Q(0)$ True. $0 \geq 0$.
2. $Q(-1)$ True. $-1 \geq -2$
3. $Q(1)$ False. $1 \geq 2$
4. $\forall x Q(x)$ False. When x=1 is a counterexample
5. $\exists x Q(x)$ True. When x=0 is an example.
6. $\exists x \neg Q(x)$ True. When x=1 is an example
7. $\forall x \neg Q(x)$ False. When x=0 is a counterexample.

Let Q(x) be the statement "$x = x^4$" and the domain consist of all integers. What are these truth values?

1. Q(0)
2. Q(1)
3. Q(2)
4. Q(-1)
5. $\forall x Q(x)$
6. $\exists x Q(x)$

Let Q(x) be the statement "$x = x^4$" and the domain consist of all integers. What are these truth values?

1. Q(0) True. 0 = 0
2. Q(1) True. 1 = 1
3. Q(2) False. 2 = 16
4. Q(-1) False. -1 = 1
5. $\forall x Q(x)$ False. When x=2 is a counterexample.
6. $\exists x Q(x)$ True. When x=0 is an example.