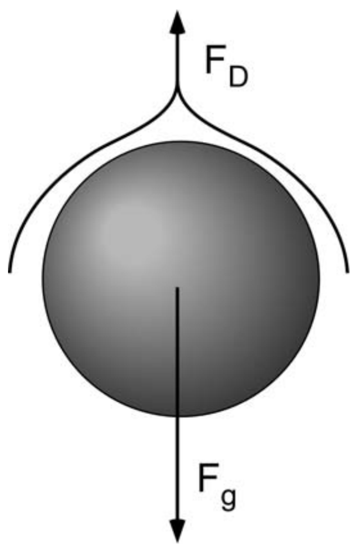
Lecture 11—Introduction to Settling Velocity

Settling velocity is one of those things that seems to have developed a whole academic *industry* around it—people have worried for a very long time how to calculate settling velocity, and how to do it accurately. This is not entirely silly; all granulometry devices that measure settling velocity as a proxy for grain size (read settling tubes) require a precise knowledge of the settling velocity of spheres, for example. Settling velocity will become a primary input for bedload transport studies, as well.

Given how important settling velocity is to sediment transport, it's not surprising that many, *many* people have taken a crack at solving this problem for once and for all. It seems so deceptively simple—consider sediment to be spherical, and balance the weight of the sphere pulling it down against the friction of water rushing past the sphere holding it up. How hard can this be?



Well, the first part is *not* hard. The weight of a sphere is just the mass times the acceleration of gravity, and the mass is just the volume times the density. SO,

$$F_g = \frac{\pi D^3}{6}\left(\rho_s - \rho_f\right)g$$

So our only *problem* is what holds the sphere back. The first formulation of this was by Stokes in 1851. The force he felt held back the sphere was the viscous resistance of the fluid on the surface of the sphere. This is related to the surface area of the sphere, the viscosity of the fluid, and the velocity of the fluid.

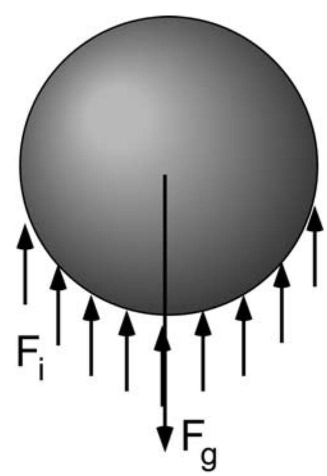
$$F_D = 3\pi D\mu v$$

To get settling velocity, just balance the two forces, and solve for v!

$$\frac{\pi D^3}{6}\left(\rho_s - \rho_f\right) = 3\pi D\mu v$$

$$v = \cancel{w} = \frac{1}{18} g \frac{\left(\rho_s - \rho_f\right)}{\mu} D^2$$

Ok, this is fine so long as viscous forces are the only ones slowing the sphere. HOWEVER, there's another force to concern ourselves with—the *impact* of water striking the sphere. Picture an extreme example. You are a particle. You are sprayed with a firehose. Are you being pushed backwards by viscosity, or by impact of water? You guessed it—riot police don't use firehoses because they spray really viscous water. So, we need a formula that handles the impact of all those little particles of water on our sphere. Consider a sphere being supported on a fountain of water:



If every particle of water strikes the sphere dead on, and fully discharges its momentum into the sphere, then the impact is related to the mass of water per unit time that strikes the sphere and the velocity. In math:

$$F_i = \frac{\pi}{4} D^2 \rho_f v^2$$

Relating *this* to the weight of the sphere gives a velocity:

$$v = \cancel{w} = \sqrt{\frac{2}{3} g \frac{\left(\rho_s - \rho_f\right)}{\rho_f} D}$$

which is, *sensu strictu*, the impact law.

Ok, there's two problems with this. One, we can't combine these two formulae yet. Two, we assumed that the particles of water all released all their momentum to the sphere, and we know for a fact that they don't—the particles along the perimeter, for example, barely graze the sphere, so why are they giving any momentum at all?

The solution to the first problem is easy. Rubey first thought this one up—the force balancing the weight of the sphere is the *combination* of the impact and the friction! Hey, who knew? Thus:

$$\frac{\pi}{6} D^3 \left(\rho_s - \rho_f\right) g = 3\pi D \mu v + \frac{\pi}{4} D^2 \rho v^2$$

and

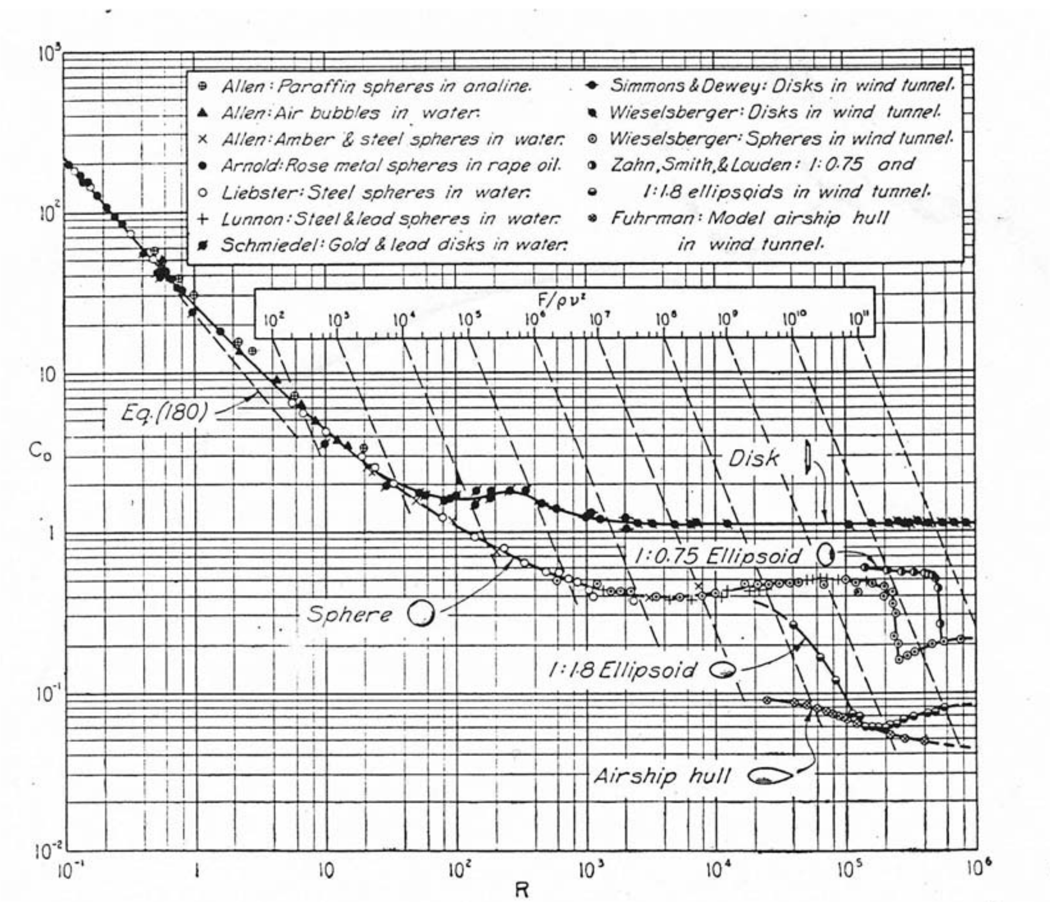
$$v = \cancel{w} = \frac{\sqrt{\frac{2}{3} g \rho_f \left(\rho_s - \rho_f\right) D^3 + 36\mu^2} - 6\mu}{\rho_f D}$$

Ok, for our second problem. We need some way of talking about how water imparts momentum to the sphere (and wouldn't it be nice if we could generalize this to things other than spheres?). One solution is to create a general force of drag, and make it account for both impact and viscous drag. Such a formulation would *look* a lot like the impact formula—you'd need the projected area, the fluid density, and the velocity:

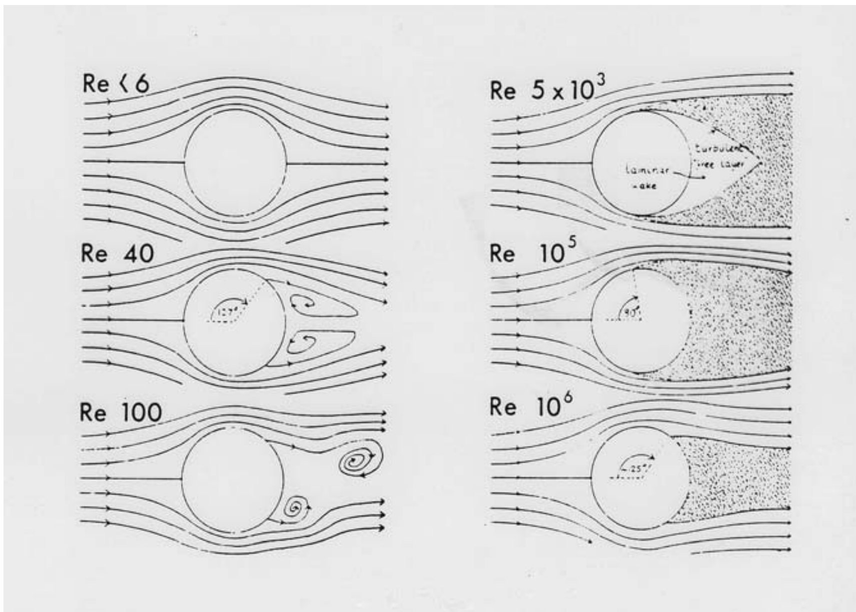
$$F_D = \frac{1}{2} C_D A \rho v^2$$

the only addition is this *drag coefficient*, that's supposed to talk about how important viscosity is, and to handle how water strikes the object (note that C_D is unitless). So, all we need is some way to talk about how water strikes an object, and how important viscosity is in all this.

For now, take it on faith—people did a lot of experiments on this and came up with a graph that defines how drag coefficient changes with changes in relative velocity. Here it is:



Notice that we have a new Reynolds number—here the velocity is the settling velocity and the length scale is particle diameter and not flow depth. This Reynolds number does basically the same thing as before—it tells us when flow around an object is laminar (and therefore friction dominated), and when flow is turbulent (and therefore impact dominated). Here's a chart that explains something about how flow behaves around a sphere for different particle Reynolds numbers:



For spheres, the plot of C_D vs. ***R*** is pretty interesting—for very low ***R***, C_D behaves as a function of ***R***, whereas for relative large ***R***, C_D becomes constant at about 0.5. This is neat—C_D shows the dominance of friction at low ***R***, and the dominance of impact at high ***R***. It also shows something called the "drag crisis" that happens to spheres at very high ***R***. Here, the particle boundary Reynolds number becomes turbulent, and the drag on the sphere suddenly drops.

Ok! So finally, then, we have a general formula for settling velocity that only has this one nasty in it.

$$v = \cancel{w} = \sqrt{\frac{4}{3}\frac{1}{C_D}gD\left(\frac{\rho_s - \rho_f}{\rho_f}\right)}$$

This is commonly referred to as "impact law," although it's a more general form than true impact law.