Lecture 12—Sensitivity Analysis and Settling Velocity

Often, we need to know how sensitive our results are to small changes (or errors) in the input variables. For situations modeled with complex computer simulations this involves varying the input parameters and observing how the output changes. When our "model" consists of an analytical expression we can calculate the sensitivity directly.

In general, for the expression:

$$y = a^{\alpha} b^{\beta} x^{\gamma}$$

y is the dependent variable, and α, β, and γ are constants. The question is, how sensitive is *y* to changes in *a*, *b*, and *x*?

We take the logarithm of both sides:

$$\log(y) = \alpha \log(a) + \beta \log(b) + \gamma \log(x)$$

and the derivative:

$$\frac{\Delta y}{y} = \alpha \frac{\Delta a}{a} + \beta \frac{\Delta b}{b} + \gamma \frac{\Delta x}{x}$$

This shows that a 10% change in the value of *a* will result in a change of
$\alpha \bullet$ 10% in the value of *y*!

We're going to use this technique on the settling velocity equation we derived, to find out how sensitive settling velocity is to various parameters (specifically, water temperature and particle diameter).

Start with the balance $F_D=F_g$:

$$\frac{1}{2} \rho C_D A w^2 = \frac{\pi D^3}{6} \left(\rho_s - \rho\right) g$$

and rearrange:

$$C_D = \frac{\frac{\pi D^3}{6}(\rho_s - \rho)g}{\frac{\pi D^2}{4}\left(\frac{1}{2}\rho w^2\right)} = \frac{4}{3}g\left(\frac{\rho_s - \rho}{\rho}\right)\frac{D}{w^2}$$

take the logarithms:

$$\log C_D = \log\left(\frac{4}{3}g\right) + \log\left(\frac{\rho_s}{\rho} - 1\right) + \log D - 2\log w$$

and derivatives:

$$\frac{d(C_D)}{C_D} = \frac{d\left(\frac{\rho_s}{\rho} - 1\right)}{\left(\frac{\rho_s}{\rho} - 1\right)} + \frac{dD}{D} - 2\frac{dw}{w}$$

since ρ_s is assumed to be constant we would like to simplify our expression to include only incremental changes in ρ_f:

$$\frac{d\left(\frac{\rho_s}{\rho} - 1\right)}{\frac{\rho_s}{\rho} - 1} = \frac{\frac{-\rho_s}{\rho^2}(d\rho)}{\frac{\rho_s}{\rho} - 1} = \frac{-\rho_s}{\rho^2}\left(\frac{d\rho}{\frac{\rho_s - \rho}{\rho}}\right) = \frac{-\rho_s}{\rho_s - \rho}\left(\frac{d\rho}{\rho}\right)$$

so,

$$\frac{dC_D}{C_D} = \frac{dD}{D} - 2\left(\frac{dw}{w}\right) - \frac{\rho_s}{\rho_s - \rho}\left(\frac{d\rho}{\rho}\right)$$

This is *almost* what we would like, but we have to resolve the incremental changes in C_D in terms of w, D, and ρ. In general, we don't have an analytical expression relating C_D to these variables…..

For *spheres*, however, we have a *graphical* relationship between C_D and Reynolds number:

$$C_D = f_1(R)$$

Here f_1 is a functional relationship between C_D and R.
The logarithm is just:

$$\log C_D = f_2(\log R)$$

and f_2 is a function just like f_1 was.

taking derivatives:

$$\frac{dC_D}{C_D} = \frac{dR}{R}\, f_2'(R)$$

But the rate of change of f_2 is just the slope of the log(C_D) vs. log(R) curve that we already have. For a particular value of C_D and R we can draw a tangent to the curve and find the slope. Let's call this "m"

Now,

$$\frac{dC_D}{C_D} = m\frac{dR}{R}$$

We *wanted* this in terms of *w*, *D*, and ρ, though. Thankfully, we can recast R in terms of these variables, and do the same analysis on them:

$$R = \frac{wD}{\nu}$$

$$\log R = \log w + \log D - \log \nu$$

$$\frac{dR}{R} = \frac{dw}{w} + \frac{dD}{D} - \frac{d\nu}{\nu}$$

substituting into our equation for C_D,

$$m\frac{dw}{w} + m\frac{dD}{D} - m\frac{d\nu}{\nu} = \frac{dD}{D} - 2\left(\frac{dw}{w}\right) - \frac{\rho_s}{\rho_s - \rho}\left(\frac{d\rho}{\rho}\right)$$

and grouping like terms:

$$\left(2+m\right)\frac{dw}{w}=\left(1-m\right)\frac{dD}{D}+m\frac{d\nu}{\nu}-\left(\frac{\rho_s}{\rho_s-\rho}\right)\frac{d\rho}{\rho}$$

and at long last:

$$\frac{dw}{w}=\frac{1-m}{2+m}\frac{dD}{D}+\frac{m}{2+m}\frac{d\nu}{\nu}-\frac{1}{2+m}\left(\frac{\rho_s}{\rho_s-\rho}\right)\frac{d\rho}{\rho}$$