MATH 11011 **FINDING REAL ZEROS OF A POLYNOMIAL** **KSU**

Definitions:

- **Polynomial**: is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

  The numbers $a_n, a_{n-1}, \ldots, a_1, a_0$ are called coefficients. a_n is the **leading coefficient**, a_0 is the constant term and the highest exponent n is the **degree** of the polynomial.

- **Zero**: If P is a polynomial and if c is a number such that $P(c) = 0$ then c is a zero of P.

Important Properties:

- The following are all equivalent:

  1. c is a zero of P.
  2. $x = c$ is an $x-$intercept of the graph of P.
  3. $x - c$ is a factor of P.
  4. $x = c$ is a solution of the equation $P(x) = 0$.

- **Rational Zeros Theorem**: If the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

  has integer coefficients, then every rational zero of P is of the form $\frac{p}{q}$ where

$$\begin{aligned} &p \text{ is a factor of the constant term } a_0, \\ \text{and } &q \text{ is a factor of the leading coefficient } a_n. \end{aligned}$$

- The Rational Zeros Theorem does NOT list irrational zeros. These will need to be found using other means like the quadratic formula.

Steps for finding the real zeros of a polynomial:

1. List all *possible* rational zeros using the Rational Zeros Theorem.

2. Use synthetic division to test the polynomial at each of the possible rational zeros that you found in step 1. (Remember that c is a zero when the remainder is zero.)

3. Repeat step 2 until you reach a quotient that is a quadratic or factors easily. Use the quadratic formula or factoring to find the remaining zeros.

Common Mistakes to Avoid:

- The Rational Zeros Theorem does NOT list the rational zeros of P. It lists all POSSIBLE rational zeros.

PROBLEMS

Find all real zeros of the polynomial.

1. $P(x) = x^3 - 7x^2 + 14x - 8$

Possible Zeros: $\pm 1,\ \pm 2,\ \pm 4,\ \pm 8.$

We start by trying 1 in synthetic division. Remember that 1 is a zero if the remainder is zero.

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 14 & -8 \\ & & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & 0 \end{array}$$

Therefore, $x = 1$ is a our first zero. Since the quotient resulting from synthetic division is always one degree less, the quotient that we have is the quadratic $x^2 - 6x + 8$ which can easily be factored.

$$x^2 - 6x + 8 = 0$$
$$(x-2)(x-4) = 0$$

Setting each factor equal to zero, we get

$$x - 2 = 0 \qquad x = 2$$

$$x - 4 = 0 \qquad x = 4$$

$$\boxed{\text{Real zeros : } \ 1,\ 2,\ 4}$$

2. $P(x) = 2x^3 + 15x^2 + 22x - 15$

Possible Zeros: $\pm 1,\ \pm 3,\ \pm 5,\ \pm 15,$
$\pm \frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{5}{2},\ \pm\frac{15}{2}.$

We start by trying -3 in synthetic division. Remember that -3 is a zero if the remainder is zero.

$$\begin{array}{r|rrrr} -3 & 2 & 15 & 22 & -15 \\ & & -6 & -27 & 15 \\ \hline & 2 & 9 & -5 & 0 \end{array}$$

Therefore, $x = -3$ is a our first zero. Since the quotient resulting from synthetic division is always one degree less, the quotient that we have is the quadratic $2x^2 + 9x - 5$ which can easily be factored.

$$2x^2 + 9x - 5 = 0$$
$$(2x-1)(x+5) = 0$$

Setting each factor equal to zero, we get

$$2x - 1 = 0 \qquad 2x = 1 \qquad x = \frac{1}{2}$$

$$x + 5 = 0 \qquad x = -5$$

$$\boxed{\text{Real zeros : } \ -3,\ -5,\ \frac{1}{2}}$$

3. $P(x) = x^3 - 22x - 15$

Possible Zeros: $\pm 1,\ \pm 3,\ \pm 5,\ \pm 15$.

We start by trying 5 in synthetic division. Remember that 5 is a zero if the remainder is zero.

$$\begin{array}{r|rrrr} 5 & 1 & 0 & -22 & -15 \\ & & 5 & 25 & 15 \\ \hline & 1 & 5 & 3 & 0 \end{array}$$

Therefore, $x = 5$ is a our first zero. Since the quotient resulting from synthetic division is always one degree less, the quotient that we have is the quadratic $x^2 + 5x + 3$ which cannot be factored. Using the quadratic formula to solve $x^2 + 5x + 3 = 0$, we get

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{25 - 12}}{2} \\ &= \frac{-5 \pm \sqrt{13}}{2} \end{aligned}$$

$$\boxed{\text{Real zeros : } \quad 5,\ \frac{-5 \pm \sqrt{13}}{2}}$$

4. $P(x) = x^3 + 3x^2 + 6x + 4$.

Possible Zeros: $\pm 1,\ \pm 2,\ \pm 4$.

We start by trying -1 in synthetic division. Remember that -1 is a zero if the remainder is zero.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 6 & 4 \\ & & -1 & -2 & -4 \\ \hline & 1 & 2 & 4 & 0 \end{array}$$

Therefore, $x = -1$ is a our first zero. Since the quotient resulting from synthetic division is always one degree less, the quotient that we have is the quadratic $x^2 + 2x + 4$ which cannot be factored. Using the quadratic formula to solve $x^2 + 2x + 4 = 0$, we get

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{4 - 16}}{2} \\ &= \frac{-5 \pm \sqrt{-12}}{2} \end{aligned}$$

Because there is a negative under the square root, these zeros are not real. Therefore, we have no further real zeros.

$$\boxed{\text{Real zeros : } \quad -1}$$

5. $P(x) = 2x^4 + x^3 - 16x^2 + 3x + 18$.

Possible Zeros: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm 9,\ \pm 18,$
$\pm \frac{1}{2}, \pm\frac{3}{2},\ \pm\frac{9}{2}$.

We start by trying 2 in synthetic division. Remember that 2 is a zero if the remainder is zero.

$$\begin{array}{r|rrrrr} 2 & 2 & 1 & -16 & 3 & 18 \\ & & 4 & 10 & -12 & -18 \\ \hline & 2 & 5 & -6 & -9 & 0 \end{array}$$

Therefore, $x = 2$ is a our first zero. However, our quotient is not a quadratic. Therefore, we will try to find one more zero by using synthetic division. We will try -1.

$$\begin{array}{r|rrrr} -1 & 2 & 5 & -6 & -9 \\ & & -2 & -3 & 9 \\ \hline & 2 & 3 & -9 & 0 \end{array}$$

Hence, $x = -1$ is our second zero. Since the quotient resulting from this synthetic division is the quadratic $2x^2 + 3x - 9$, we will factor this to solve.

$$2x^2 + 3x - 9 = 0$$
$$(2x - 3)(x + 3) = 0$$

Setting each factor equal to zero, we get

$$2x - 3 = 0 \qquad\qquad x + 3 = 0$$
$$2x = 3 \qquad\qquad x = -3$$
$$x = \frac{3}{2}$$

$$\boxed{\text{Real zeros : } \quad 2,\ -1,\ -3,\ \frac{3}{2}}$$

6. $P(x) = 2x^4 + 7x^3 - x^2 - 15x - 9$

Possible Zeros: $\pm 1,\ \pm 3,\ \pm 9,$
$\pm \frac{1}{2}, \pm\frac{3}{2},\ \pm\frac{9}{2}$.

We start by trying -1 in synthetic division. Remember that -1 is a zero if the remainder is zero.

$$\begin{array}{r|rrrrr} -1 & 2 & 7 & -1 & -15 & -9 \\ & & -2 & -5 & 6 & 9 \\ \hline & 2 & 5 & -6 & -9 & 0 \end{array}$$

Therefore, $x = -1$ is our first zero. However, our quotient is not a quadratic. Therefore, we will try to find one more zero by using synthetic division. We will try -3.

$$\begin{array}{r|rrrr} -3 & 2 & 5 & -6 & -9 \\ & & -6 & 3 & 9 \\ \hline & 2 & -1 & -3 & 0 \end{array}$$

Hence, $x = -3$ is our second zero. Since the quotient resulting from this synthetic division is the quadratic $2x^2 - x - 3$, we will factor this to solve.

$$2x^2 - x - 3 = 0$$
$$(2x - 3)(x + 1) = 0$$

Setting each factor equal to zero, we get

$$2x - 3 = 0 \qquad\qquad x + 1 = 0$$
$$2x = 3 \qquad\qquad x = -1$$
$$x = \frac{3}{2}$$

$$\boxed{\text{Real zeros : } \quad -1,\ -3,\ \frac{3}{2}}$$

NOTE: $x = -1$ is a double zero. It occurs two times. However, you only need to list it once.

7. $P(x) = 6x^4 - 8x^3 - 41x^2 + 23x + 30.$

Possible Zeros: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 5,\ \pm 6,$

$$\pm\, 10,\ \pm 15, \pm 30$$
$$\pm\, \tfrac{1}{2}, \pm\tfrac{3}{2},\ \pm\tfrac{5}{2},$$
$$\pm\, \tfrac{1}{3},\ \pm\tfrac{2}{3},\ \pm\tfrac{5}{3},\ \pm\tfrac{10}{3},$$
$$\pm\, \tfrac{1}{6},\ \pm\tfrac{5}{6}$$

We start by trying 3 in synthetic division. Remember that 3 is a zero if the remainder is zero.

$$\begin{array}{r|rrrrr} 3 & 6 & -8 & -41 & 23 & 30 \\ & & 18 & 30 & -33 & -30 \\ \hline & 6 & 10 & -11 & -10 & 0 \end{array}$$

Therefore, $x = 3$ is a our first zero. However, our quotient is not a quadratic. Therefore, we will try to find one more zero by using synthetic division. We will try $-\frac{2}{3}$.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 6 & 10 & -11 & -10 \\ & & -4 & -4 & 10 \\ \hline & 6 & 6 & -15 & 0 \end{array}$$

Hence, $x = -\frac{2}{3}$ is our second zero. Since the quotient resulting from this synthetic division is the quadratic $6x^2 + 6x - 15$, we need to solve this.

$$6x^2 + 6x - 15 = 0$$
$$3(2x^2 + 2x - 5) = 0$$

Using the quadratic formula to solve $2x^2 + 2x - 5 = 0$, we get

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(2)(-5)}}{2(2)} \\ &= \frac{-2 \pm \sqrt{4 + 40}}{4} \\ &= \frac{-2 \pm \sqrt{44}}{4} \\ &= \frac{-2 \pm 2\sqrt{11}}{4} \\ &= \frac{2(-1 \pm \sqrt{11})}{4} \\ &= \frac{-1 \pm \sqrt{11}}{2} \end{aligned}$$

$$\boxed{\text{Real zeros : } \quad 3,\ -\frac{2}{3},\quad \frac{-1 \pm \sqrt{11}}{2}}$$