Definitions:

- **Rectangular Coordinate System**: consists of a vertical line called the $y-$axis and a horizontal line called the $x-$axis. The $x-$axis and $y-$axis divide the coordinate plane into four quadrants and intersect at a point called the **origin**. Each point in the plane corresponds to a unique ordered pair (x, y).
- **Midpoint**: of a line segment AB is the point that is equidistant from the endpoints A and B.

Important Properties:

- **Distance formula**: Suppose that $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are two points in the coordinate plane. The distance between A and B, denoted $d(A, B)$, is given by

  $$\boxed{d(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.}$$

  It does not matter in what order you subtract the $x-$coordinates or the $y-$coordinates.
- **Midpoint formula**: Suppose that $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are the endpoints of the line segment AB. Then the midpoint M of AB is given by

  $$\boxed{M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).}$$

- **Pythagorean Theorem**: In a right triangle, if the side opposite the right angle has length c and the other two sides have lengths a and b, then

  $$\boxed{a^2 + b^2 = c^2.}$$

  The side opposite the right angle is called the **hypotenuse** and the other two sides are called **legs**.
- The converse of the Pythagorean Theorem is also true. Namely, if a triangle has sides of lengths a, b and c which satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Common Mistakes to Avoid:

- The midpoint is found by *averaging* the $x-$coordinates and *averaging* the $y-$coordinates. Do NOT subtract them.
- The square root of a sum is NOT the sum of the square roots. In other words,

  $$\sqrt{x^2 + y^2} \neq \sqrt{x^2} + \sqrt{y^2}.$$

  To illustrate this with an example, notice that

  $$\sqrt{9 + 16} = \sqrt{25} = 5 \qquad \text{whereas} \qquad \sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

- When using the Pythagorean Theorem, make sure that the hypotenuse is on a side by itself; namely,

  $$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2.$$

PROBLEMS

1. **Find the distance between the given points.**

(a) $A = (2, 0)$ and $B = (0, 9)$

$$\begin{aligned} d(A, B) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(2 - 0)^2 + (0 - 9)^2} \\ &= \sqrt{(2)^2 + (-9)^2} \\ &= \sqrt{4 + 81} \\ &= \sqrt{85} \end{aligned}$$

$$\boxed{d(A, B) = \sqrt{85}}$$

(b) $A = (-2, 5)$ and $B = (12, 3)$

$$\begin{aligned} d(A, B) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(-2 - 12)^2 + (5 - 3)^2} \\ &= \sqrt{(-14)^2 + (2)^2} \\ &= \sqrt{196 + 4} \\ &= \sqrt{200} \\ &= \sqrt{100}\sqrt{2} \\ &= 10\sqrt{2} \end{aligned}$$

$$\boxed{d(A, B) = 10\sqrt{2}}$$

(c) $A = (-2, 3)$ and $B = (9, -3)$

$$\begin{aligned} d(A, B) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(-2 - 9)^2 + (3 - (-3))^2} \\ &= \sqrt{(-11)^2 + (6)^2} \\ &= \sqrt{121 + 36} \\ &= \sqrt{157} \end{aligned}$$

$$\boxed{d(A, B) = \sqrt{157}}$$

(d) $A = (-1, -5)$ and $B = (-4, 7)$

$$\begin{aligned} d(A, B) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(-1 - (-4))^2 + (-5 - 7)^2} \\ &= \sqrt{(3)^2 + (-12)^2} \\ &= \sqrt{9 + 144} \\ &= \sqrt{153} \\ &= \sqrt{9}\sqrt{17} \\ &= 3\sqrt{17} \end{aligned}$$

$$\boxed{d(A, B) = 3\sqrt{17}}$$

2. **Find the midpoint M of the line segment AB where**

(a) $A = (8, -4)$ and $B = (-2, 2)$

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{8 + -2}{2}, \frac{-4 + 2}{2}\right) \\ &= \left(\frac{6}{2}, \frac{-2}{2}\right) \\ &= (3, -1) \end{aligned}$$

$$\boxed{M = (3, -1)}$$

(b) $A = (5, -6)$ and $B = (-2, 11)$

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{5 + -2}{2}, \frac{-6 + 11}{2}\right) \\ &= \left(\frac{3}{2}, \frac{5}{2}\right) \end{aligned}$$

$$\boxed{M = \left(\frac{3}{2}, \frac{5}{2}\right)}$$

(c) $A = (8, 2)$ and $B = \left(\frac{1}{2}, -\frac{1}{2}\right)$

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{8 + \frac{1}{2}}{2}, \frac{2 + -\frac{1}{2}}{2}\right) \\ &= \left(\frac{\frac{17}{2}}{2}, \frac{\frac{3}{2}}{2}\right) \\ &= \left(\frac{17}{4}, \frac{3}{4}\right) \end{aligned}$$

$$\boxed{M = \left(\frac{17}{4}, \frac{3}{4}\right)}$$

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3. **If $M = (3, -2)$ is the midpoint of the line segment AB and if $A = (-9, 2)$ find the coordinates of B.**

Let $B = (x, y)$. Then using the midpoint formula we get

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ (3, -2) &= \left(\frac{x + (-9)}{2}, \frac{y + 2}{2}\right) \end{aligned}$$

Therefore, equating coordinates, we find that

$$\begin{aligned} \frac{x + (-9)}{2} &= 3 \\ x + (-9) &= 6 \\ x &= 15 \end{aligned}$$

$$\begin{aligned} \frac{y + 2}{2} &= -2 \\ y + 2 &= -4 \\ y &= -6 \end{aligned}$$

$$\boxed{B = (15, -6)}$$

4. **Find the point on the line segment AB that is one-fourth of the distance from the point $A = (3, -4)$ to the point $B = (-5, 12)$.**

First, we will find the midpoint of the line segment.

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3 + -5}{2}, \frac{-4 + 12}{2}\right) \\ &= \left(\frac{-2}{2}, \frac{8}{2}\right) \\ &= (-1, 4) \end{aligned}$$

Now, M is one-half the distance from A to B. Therefore, the point we need, lets call it C, is the midpoint between A and M.

$$\begin{aligned} C &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3 + -1}{2}, \frac{-4 + 4}{2}\right) \\ &= \left(\frac{2}{2}, \frac{0}{2}\right) \\ &= (1, 0) \end{aligned}$$

$$\boxed{C = (1, 0)}$$

5. **Determine the point C on the $x-$axis that is equidistant from $A = (-2, 3)$ and $B = (1, -5)$.**

NOTE: Although we are looking for the point equidistant, we are NOT looking for the midpoint. Since we are told the point lies on the $x-$axis, we are looking for a point of the form $C = (x, 0)$. Since they are equidistant, they must have the same distances. Therefore, we will equate the distance formulas.

$$\begin{aligned} d(A, C) &= d(B, C) \\ \sqrt{(-2 - x)^2 + (3 - 0)^2} &= \sqrt{(1 - x)^2 + (-5 - 0)^2} \\ (-2 - x)^2 + (3)^2 &= (1 - x)^2 + (-5)^2 \\ 4 + 4x + x^2 + 9 &= 1 - 2x + x^2 + 25 \\ x^2 + 4x + 13 &= x^2 - 2x + 26 \\ 4x + 13 &= -2x + 26 \\ 6x + 13 &= 26 \\ 6x &= 13 \\ x &= \frac{13}{6} \end{aligned}$$

$$\boxed{C = \left(\frac{13}{6}, 0\right)}$$

6. **Determine if $A = (2, -3)$, $B = (1, 8)$ and $C = (-4, 2)$ are the vertices of a right triangle.**

To solve this, we will first find the distances between the three points. Then we will check to see if they satisfy the Pythagorean Theorem.

$$\begin{aligned} d(A,B) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(2-1)^2 + (-3-8)^2} \\ &= \sqrt{(1)^2 + (-11)^2} \\ &= \sqrt{1 + 121} \\ &= \sqrt{122} \end{aligned}$$

$$\begin{aligned} d(A,C) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(2-(-4))^2 + (-3-2)^2} \\ &= \sqrt{(6)^2 + (-5)^2} \\ &= \sqrt{36 + 25} \\ &= \sqrt{61} \end{aligned}$$

$$\begin{aligned} d(B,C) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(1-(-4))^2 + (8-2)^2} \\ &= \sqrt{(5)^2 + (6)^2} \\ &= \sqrt{25 + 36} \\ &= \sqrt{61} \end{aligned}$$

Now, substituting into $a^2 + b^2 = c^2$, we find

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (\sqrt{61})^2 + (\sqrt{61})^2 &= (\sqrt{122})^2 \\ 61 + 61 &= 122 \\ 122 &= 122\star \end{aligned}$$

Therefore, by the converse of the Pythagorean Theorem,

$$\boxed{\text{Triangle } ABC \text{ is a right triangle.}}$$