Definition:

- **Quadratic-Type Expression:** is an expression of the form $au^2 + bu + c$ where u is an algebraic expression. For example, the following are all quadratic-type expressions:

$$x^4 + 5x^2 + 6, \qquad (x-3)^2 - 2(x-3) - 3, \qquad x^8 - 7x^4 - 18.$$

Important Properties:

- **Zero Product Property**: If a and b are real numbers and $a \cdot b = 0$, then $a = 0$ or $b = 0$.
- If you recognize how the expression factors in its original form, then do so. If not, then use a u substitution to create a quadratic that is easier to factor. For example, for $x^4 + 5x^2 + 6$, if we let $u = x^2$ then we have

$$x^4 + 5x^2 + 6 = u^2 + 5u + 6 = (u+3)(u+2) = (x^2+2)(x^2+3).$$

- Only use the substitution if it helps you factor the expression.

Common Mistakes to Avoid:

- If you use the u substitution, you must substitute back to the original variable. So, if the original problem is in terms of x, make sure that your answer is also in terms of x.
- Although you can raise both **sides** of an equation to the same power without changing the solutions, you can NOT raise each **term** to the same power.
- Remember that whenever you have the even root of a positive number, we get two answers: one positive and one negative. For example, if $x^4 = 16$ then by taking the 4th root of both sides we get $x = 2$ AND $x = -2$. Do NOT forget the negative answer when working with even roots.
- Do NOT attach a $\pm$ when working with odd roots. When you take the odd root of a number, you get only one solution.
- Make sure that the variable is isolated before raising both sides to the same power. For example,

$$\left(4x^{2/3}\right)^{3/2} \neq 4x.$$

PROBLEMS

Solve for x in each of the following equations.

1. $x^4 + 15x^2 - 16 = 0$

$$x^4 + 15x^2 - 16 = 0$$
$$(x^2 - 1)(x^2 + 16) = 0$$
$$(x - 1)(x + 1)(x^2 + 16) = 0$$

Setting each factor equal to zero, we get

$$x - 1 = 0 \qquad x = 1$$

$$x + 1 = 0 \qquad x = -1$$

$$x^2 + 16 = 0 \qquad x^2 \neq -16$$

$$\boxed{x = 1, \qquad x = -1}$$

OR (for an alternative way)
Letting $u = x^2$, we get

$$x^4 + 15x^2 - 16 = 0$$
$$u^2 + 15u - 16 = 0$$
$$(u + 16)(u - 1) = 0$$

Setting each factor equal to zero, we get

$$u + 16 = 0 \qquad u = -16 \qquad x^2 \neq -16$$

$$u - 1 = 0 \qquad u = 1 \qquad x^2 = 1 \qquad \sqrt{x^2} = \sqrt{1} \qquad x = \pm 1$$

$$\boxed{x = 1, \qquad x = -1}$$

2. $9x^4 - 14x^2 + 5 = 0$

$$9x^4 - 14x^2 + 5 = 0$$
$$(9x^2 - 5)(x^2 - 1) = 0$$

Setting each factor equal to zero, we get

$$9x^2 - 5 = 0$$
$$9x^2 = 5$$
$$x^2 = \frac{5}{9}$$
$$\sqrt{x^2} = \sqrt{\frac{5}{9}}$$
$$x = \pm\frac{\sqrt{5}}{3}$$

$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$\sqrt{x^2} = \sqrt{1}$$
$$x = \pm 1$$

$$\boxed{x = \pm\frac{\sqrt{5}}{3}, \qquad x = \pm 1}$$

OR (for an alternative way)

Letting $u = x^2$, we get

$$9x^4 - 14x^2 + 5 = 0$$
$$9u^2 - 14u + 5 = 0$$
$$(9u - 5)(u - 1) = 0$$

Setting each factor equal to zero, we get

$$9u - 5 = 0$$
$$9u = 5$$
$$u = \frac{5}{9}$$
$$x^2 = \frac{5}{9}$$
$$\sqrt{x^2} = \sqrt{\frac{5}{9}}$$
$$x = \pm\frac{\sqrt{5}}{3}$$

$$u - 1 = 0$$
$$u = 1$$
$$x^2 = 1$$
$$\sqrt{x^2} = \sqrt{1}$$
$$x = \pm 1$$

$$\boxed{x = \pm\frac{\sqrt{5}}{3}, \qquad x = \pm 1}$$

3. $x^{-2} - 9x^{-1} - 10 = 0$

$$x^{-2} - 9x^{-1} - 10 = 0$$
$$\left(x^{-1} - 10\right)\left(x^{-1} + 1\right) = 0$$

Setting each factor equal to zero, we get

$$x^{-1} - 10 = 0$$
$$x^{-1} = 10$$
$$\frac{1}{x} = 10$$
$$x = \frac{1}{10}$$

$$x^{-1} + 1 = 0$$
$$x^{-1} = -1$$
$$\frac{1}{x} = -1$$
$$x = -1$$

$$\boxed{x = \frac{1}{10}, \qquad x = -1}$$

OR (for an alternative way)
Letting $u = x^{-1}$, we obtain

$$x^{-2} - 9x^{-1} - 10 = 0$$
$$u^2 - 9u - 10 = 0$$
$$(u - 10)(u + 1) = 0$$

Setting each factor equal to zero, we get

$$u - 10 = 0$$
$$u = 10$$
$$x^{-1} = 10$$
$$\frac{1}{x} = 10$$
$$x = \frac{1}{10}$$

$$u + 1 = 0$$
$$u = -1$$
$$x^{-1} = -1$$
$$\frac{1}{x} = -1$$
$$x = -1$$

$$\boxed{x = \frac{1}{10}, \qquad x = -1}$$

4. $x^{2/3} - 2x^{1/3} - 8 = 0$

$$x^{2/3} - 2x^{1/3} - 8 = 0$$
$$\left(x^{1/3} - 4\right)\left(x^{1/3} + 2\right) = 0$$

Setting each factor equal to zero, we obtain

$$x^{1/3} - 4 = 0$$
$$x^{1/3} = 4$$
$$\left(x^{1/3}\right)^3 = 4^3$$
$$x = 64$$

$$x^{1/3} + 2 = 0$$
$$x^{1/3} = -2$$
$$\left(x^{1/3}\right)^3 = (-2)^3$$
$$x = -8$$

$$\boxed{x = 64, \qquad x = -8}$$

OR (for an alternative way)
Letting $u = x^{1/3}$, we get

$$x^{2/3} - 2x^{1/3} - 8 = 0$$
$$u^2 - 2u - 8 = 0$$
$$(u - 4)(u + 2) = 0$$

Setting each factor equal to zero, we obtain

$$u - 4 = 0$$
$$u = 4$$
$$x^{1/3} = 4$$
$$\left(x^{1/3}\right)^3 = 4^3$$
$$x = 64$$

$$u + 2 = 0$$
$$u = -2$$
$$x^{1/3} = -2$$
$$\left(x^{1/3}\right)^3 = (-2)^3$$
$$x = -8$$

$$\boxed{x = 64, \qquad x = -8}$$

5. $x^{1/2} + 6 = 5x^{1/4}$

First, we will move everything to one side and factor completely.

$$x^{1/2} + 6 = 5x^{1/4}$$
$$x^{1/2} - 5x^{1/4} + 6 = 0$$
$$\left(x^{1/4} - 3\right)\left(x^{1/4} - 2\right) = 0$$

Setting each factor to zero, we get

$$x^{1/4} - 3 = 0$$
$$x^{1/4} = 3$$
$$\left(x^{1/4}\right)^4 = 3^4$$
$$x = 81$$

$$x^{1/4} - 2 = 0$$
$$x^{1/4} = 2$$
$$\left(x^{1/4}\right)^4 = 2^4$$
$$x = 16$$

Because we raised both sides to an even power, we must check our answers in the original equation.

Checking: $x = 81$

$$(81)^{1/2} + 6 = 5(81)^{1/4}$$
$$\sqrt{81} + 6 = 5\sqrt[4]{81}$$
$$9 + 6 = 5(3)$$
$$15 = 15\star$$

Checking: $x = 16$

$$(16)^{1/2} + 6 = 5(16)^{1/4}$$
$$\sqrt{16} + 6 = 5\sqrt[4]{16}$$
$$4 + 6 = 5(2)$$
$$10 = 10\star$$

$$\boxed{x = 81, \qquad x = 16}$$

6. $x^{1/3} + 2x^{1/6} = 3$

We first need to move everything to one side before we can factor.

$$x^{1/3} + 2x^{1/6} = 3$$
$$x^{1/3} + 2x^{1/6} - 3 = 0$$
$$\left(x^{1/6} + 3\right)\left(x^{1/6} - 1\right) = 0$$

Setting each factor equal to zero, we get

$$x^{1/6} + 3 = 0$$
$$x^{1/6} = -3$$
$$\left(x^{1/6}\right)^6 = (-3)^6$$
$$x = 729$$

$$x^{1/6} - 1 = 0$$
$$x^{1/6} = 1$$
$$\left(x^{1/6}\right)^6 = (1)^6$$
$$x = 1$$

Because we raised both sides to an even power, we need to check our answers in the original equation.

<u>Check: $x = 729$</u>

$$\sqrt[3]{729} + 2\sqrt[6]{729} = 3$$
$$9 + 2(3) = 3$$
$$15 \neq 3$$

<u>Check: $x = 1$</u>

$$(1)^{1/3} + 2(1)^{1/6} = 3$$
$$1 + 2 = 3$$
$$3 = 3\star$$

Since $x = 729$ does not check, it cannot be a solution.

$$\boxed{x = 1}$$

OR (for an alternative way)
Letting $u = x^{1/6}$, we get

$$x^{1/3} + 2x^{1/6} = 3$$
$$x^{1/3} + 2x^{1/6} - 3 = 0$$
$$u^2 + 2u - 3 = 0$$
$$(u + 3)(u - 1) = 0$$

Setting each factor equal to zero, we get

$$\begin{aligned} u+3&=0\\ u&=-3\\ x^{1/6}&=-3\\ \left(x^{1/6}\right)^6&=(-3)^6\\ x&=729 \end{aligned}$$

$$\begin{aligned} u-1&=0\\ u&=1\\ x^{1/6}&=1\\ \left(x^{1/6}\right)^6&=(1)^6\\ x&=1 \end{aligned}$$

Once again, we need to check our answers since we raised both sides to an even power. See check above.

$$\boxed{x=1}$$

---

7. $\left(3-x^{1/2}\right)^2-10\left(3-x^{1/2}\right)+21=0$

Notice that this is a quadratic-type equation with $u=3-x^{1/2}$. Therefore,

$$\begin{aligned} \left(3-x^{1/2}\right)^2-10\left(3-x^{1/2}\right)+21&=0\\ u^2-10u+21&=0\\ (u-7)(u-3)&=0 \end{aligned}$$

Setting each factor equal to zero, we get

$$\begin{aligned} u-7&=0\\ u&=7\\ 3-x^{1/2}&=7\\ -x^{1/2}&=4\\ x^{1/2}&=-1\\ \left(x^{1/2}\right)^2&=(-1)^2\\ x&=1 \end{aligned}$$

$$\begin{aligned} u-3&=0\\ u&=3\\ 3-x^{1/2}&=3\\ -x^{1/2}&=0\\ x^{1/2}&=0\\ \left(x^{1/2}\right)^2&=0^2\\ x&=0 \end{aligned}$$

Since we raised both sides to an even power, we must check our answers in the original equation.

Check: $x = 1$

$$\left(3 - 1^{1/2}\right)^2 - 10\left(3 - 1^{1/2}\right) + 21 = 0$$
$$(3-1)^2 - 10(3-1) + 21 = 0$$
$$2^2 - 10(2) + 21 = 0$$
$$4 - 20 + 21 = 0$$
$$5 \neq 0$$

Check: $x = 0$

$$\left(3 - 0^{1/2}\right)^2 - 10\left(3 - 0^{1/2}\right) + 21 = 0$$
$$(3-0)^2 - 10(3-0) + 21 = 0$$
$$3^2 - 10(3) + 21 = 0$$
$$9 - 30 + 21 = 0$$
$$0 = 0\star$$

Since $x = 1$ does not check in the original equation it cannot be a solution.

$$\boxed{x = 0}$$