SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

Definition:

- **Quadratic Equation**: is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers, $a \neq 0$.

Steps for solving a quadratic equation by completing the square :

1. Isolate the constant on one side.
2. Make sure that the coefficient in front of the squared term is a *positive* one. If it is not, divide every term of the equation by this coefficient.
3. Once the coefficient on the squared term is a positive one, take one-half the coefficient of x and square this result. Namely, $\left(\frac{1}{2} \cdot \text{coeff of } x\right)^2$.
4. Add the result of Step 3 to both sides.
5. Factor the side containing the variables as a perfect square.
6. Solve the resulting equation using the square root property.

Important Properties:

- **Square Root Property**: If c is a positive number and if $x^2 = c$, then

$$x = \sqrt{c} \qquad \text{or} \qquad x = -\sqrt{c}.$$

(This can be written in one statement as $\pm\sqrt{c}$.) In other words, when solving a quadratic equation by the square root property, we want both the positive and negative square roots.

- Completing the square can be used to solve *any* quadratic equation.

Common Mistakes to Avoid:

- While completing the square, do not proceed to step three until the coefficient on the squared term is a POSITIVE one.
- Do NOT forget the $\pm$ when taking square roots.
- If the solution can be simplified, you must do so. For example, $x = -5 \pm 2$ becomes $x = -5 + 2 = -3$ and $x = -5 - 2 = -7$.

PROBLEMS

Solve the following equations by completing the square.

1. $x^2 + 4x = 5$

$$\begin{aligned} x^2 + 4x &= 5 \\ \left(\frac{1}{2} \cdot 4\right)^2 &= (2)^2 = 4 \\ x^2 + 4x + 4 &= 5 + 4 \\ (x+2)^2 &= 9 \\ \sqrt{(x+2)^2} &= \sqrt{9} \\ x + 2 &= \pm 3 \\ x &= -2 \pm 3 \\ x &= -2 + 3 = 1 \\ x &= -2 - 3 = -5 \end{aligned}$$

$$\boxed{x = 1, \quad x = -5}$$

2. $x^2 + 4x + 1 = 0$

$$\begin{aligned} x^2 + 4x + 1 &= 0 \\ x^2 + 4x &= -1 \\ \left(\frac{1}{2} \cdot 4\right)^2 &= (2)^2 = 4 \\ x^2 + 4x + 4 &= -1 + 4 \\ (x+2)^2 &= 3 \\ \sqrt{(x+2)^2} &= \sqrt{3} \\ x + 2 &= \pm\sqrt{3} \\ x &= -2 \pm \sqrt{3} \end{aligned}$$

$$\boxed{x = -2 + \sqrt{3}, \quad x = -2 - \sqrt{3}}$$

3. $x^2 - 8x - 5 = 0$

$$\begin{aligned} x^2 - 8x - 5 &= 0 \\ x^2 - 8x &= 5 \\ \left(\frac{1}{2} \cdot -8\right)^2 &= (-4)^2 = 16 \\ x^2 - 8x + 16 &= 5 + 16 \\ (x-4)^2 &= 21 \\ \sqrt{(x-4)^2} &= \sqrt{21} \\ x - 4 &= \pm\sqrt{21} \\ x &= 4 \pm \sqrt{21} \end{aligned}$$

$$\boxed{x = 4 + \sqrt{21}, \quad x = 4 - \sqrt{21}}$$

4. $2x^2 - 12x + 14 = 0$

$$\begin{aligned} 2x^2 - 12x + 14 &= 0 \\ 2x^2 - 12x &= -14 \\ x^2 - 6x &= -7 \\ \left(\frac{1}{2} \cdot -6\right)^2 &= (-3)^2 = 9 \\ x^2 - 6x + 9 &= -7 + 9 \\ (x-3)^2 &= 2 \\ \sqrt{(x-3)^2} &= \sqrt{2} \\ x - 3 &= \pm\sqrt{2} \\ x &= 3 \pm \sqrt{2} \end{aligned}$$

$$\boxed{x = 3 + \sqrt{2}, \quad x = 3 - \sqrt{2}}$$

5. $4x^2 + 32x - 3 = 0$

$$4x^2 + 32x - 3 = 0$$
$$4x^2 + 32x = 3$$
$$x^2 + 8x = \frac{3}{4}$$
$$\left(\frac{1}{2} \cdot 8\right)^2 = (4)^2 = 16$$
$$x^2 + 8x + 16 = \frac{3}{4} + 16$$
$$(x+4)^2 = \frac{69}{4}$$
$$\sqrt{(x+4)^2} = \sqrt{\frac{69}{4}}$$
$$x + 4 = \pm\frac{\sqrt{69}}{2}$$
$$x = -4 \pm \frac{\sqrt{69}}{2}$$

$$\boxed{x = -4 + \frac{\sqrt{69}}{2}, \quad x = -4 - \frac{\sqrt{69}}{2}}$$

6. $5x^2 + 5x - 15 = 0$

$$5x^2 + 5x - 15 = 0$$
$$5x^2 + 5x = 15$$
$$x^2 + x = 3$$
$$\left(\frac{1}{2} \cdot 1\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$
$$x^2 + x + \frac{1}{4} = 3 + \frac{1}{4}$$
$$\left(x + \frac{1}{2}\right)^2 = \frac{13}{4}$$
$$\sqrt{\left(x + \frac{1}{2}\right)^2} = \sqrt{\frac{13}{4}}$$
$$x + \frac{1}{2} = \pm\frac{\sqrt{13}}{2}$$
$$x = -\frac{1}{2} \pm \frac{\sqrt{13}}{2}$$

$$\boxed{x = -\frac{1}{2} + \frac{\sqrt{13}}{2}, \quad x = -\frac{1}{2} - \frac{\sqrt{13}}{2}}$$

7. $3x^2 + 7x = 4$

$$
\begin{aligned}
3x^2 + 7x &= 4 \\
x^2 + \frac{7}{3}x &= \frac{4}{3} \\
\left(\frac{1}{2} \cdot \frac{7}{3}\right)^2 &= \left(\frac{7}{6}\right)^2 = \frac{49}{36} \\
x^2 + \frac{7}{3}x + \frac{49}{36} &= \frac{4}{3} + \frac{49}{36} \\
x^2 + \frac{7}{3}x + \frac{49}{36} &= \frac{48}{36} + \frac{49}{36} \\
x^2 + \frac{7}{3}x + \frac{49}{36} &= \frac{97}{36} \\
\left(x + \frac{7}{6}\right)^2 &= \frac{97}{36} \\
\sqrt{\left(x + \frac{7}{6}\right)^2} &= \sqrt{\frac{97}{36}} \\
x + \frac{7}{6} &= \pm\frac{\sqrt{97}}{6} \\
x &= -\frac{7}{6} \pm \frac{\sqrt{97}}{6}
\end{aligned}
$$

$$
\boxed{x = -\frac{7}{6} + \frac{\sqrt{97}}{6}, \quad x = -\frac{7}{6} - \frac{\sqrt{97}}{6}}
$$