

Chapter 13

General Relativity

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This will be somewhat of a strange chapter, because we won't have enough time to get to the heart of General Relativity (GR). But we will still be able to get a flavor of the subject and derive a few interesting GR results.

One crucial idea in GR is the Equivalence Principle. This basically says that gravity is equivalent to acceleration. We will have much to say about this issue in the sections below. Another crucial concept in GR is that of coordinate independence. The laws of physics should not depend on what coordinate system you choose. This seemingly innocuous statement has surprisingly far-reaching consequences. However, a discussion of this topic is one of the many things we won't have time for. We would need a whole class on GR to do it justice. But fortunately, it is possible to get a sense of the nature of GR without having to master such things. This is the route we will take in this chapter.

13.1 The Equivalence Principle

Einstein's Equivalence Principle says that it is impossible to locally distinguish between gravity and acceleration. This may be stated more precisely in (at least) three ways.

- Let person A be enclosed in a small box, far from any massive objects, that undergoes uniform acceleration (say, g). Let person B stand at rest on the earth (see Fig. 13.1). The Equivalence Principle says that there are no local experiments these two people can perform that will tell them which of the two settings they are in. The physics of each setting is the same.
- Let person A be enclosed in a small box that is in free-fall near a planet. Let person B float freely in space, far away from any massive objects (see Fig. 13.2). The Equivalence Principle says that there are no local experiments these two people can perform that will tell them which of the two settings they are in. The physics of each setting is the same.
- “Gravitational” mass is equal to (or proportional to) “inertial” mass. Gravitational mass is the m_g that appears in the formula, $F = GMm_g/r^2 \equiv m_g g$.

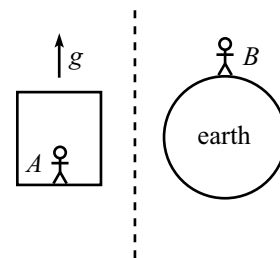


Figure 13.1

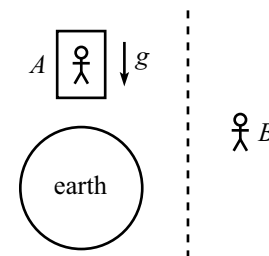


Figure 13.2

Inertial mass is the m_i that appears in the formula, $F = m_i a$. There is no *a priori* reason why these two m 's should be the same (or proportional). An object that is dropped on the earth will have acceleration $a = (m_g/m_i)g$. For all we know, the ratio m_g/m_i for plutonium is different from that for copper. But experiments with various materials have detected no difference in the ratios. The Equivalence Principle states that the ratios are equal for any type of mass.

This definition of the Equivalence Principle is equivalent to, say, the second one above for the following reason. Two different masses near B will stay right where they are. But two different masses near A will diverge from each other if their accelerations are not equal.

These statements are all quite believable. Consider the first one, for example. When standing on the earth, you have to keep your legs firm to avoid falling down. When standing in the accelerating box, you have to keep your legs firm to maintain the same position relative to the floor (that is, to avoid “falling down”). You certainly can't naively tell the difference between the two scenarios. The Equivalence Principle says that it's not just that you're too inept to figure out a way to differentiate between them, but instead that there is no possible local experiment you can perform to tell the difference, no matter how clever you are.

REMARK: Note the inclusion of the words “small box” and “local” above. On the surface of the earth, the lines of the gravitational force are not parallel; they converge to the center. The gravitational force also varies with height. Therefore, an experiment performed over a non-negligible distance (for example, dropping two balls next to each other, and watching them converge; or dropping two balls on top of each other and watching them diverge) will have different results from the same experiment in the accelerating box. The equivalence principle says that if your laboratory is small enough, or if the gravitational field is sufficiently uniform, then the two scenarios look essentially the same. ♣

13.2 Time dilation

The equivalence principle has a striking consequence concerning the behavior of clocks in a gravitational field. It implies that higher clocks run faster than lower clocks. If you put a watch on top of a tower, and then stand on the ground, you will see the watch on the tower tick faster than an identical watch on your wrist. When you take the watch down and compare it to the one on your wrist, it will show more time elapsed.¹ Likewise, someone standing on top of the tower will see a clock on the ground run slow. Let's be quantitative about this. Consider the following two scenarios.

¹This will be true only if you keep the watch on the tower for a long enough time, because the movement of the watch will cause it to run slow due to the usual special-relativistic time dilation. But the (speeding-up) effect due to the height can be made arbitrarily large compared to the (slowing-down) effect due to the motion, by simply keeping the watch on the tower for an arbitrarily long time.

- A light source on top of a tower of height h emits flashes at time intervals t_s . A receiver on the ground receives the flashes at time intervals t_r (see Fig. 13.3). What is t_r in terms of t_s ?
- A rocket of length h accelerates with acceleration g . A light source at the front end emits flashes at time intervals t_s . A receiver at the back end receives the flashes at time intervals t_r (see Fig. 13.4). What is t_r in terms of t_s ?

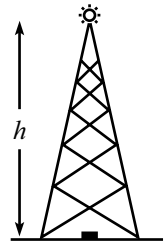


Figure 13.3

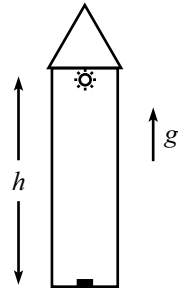


Figure 13.4

The equivalence principle tells us that these two scenarios look exactly the same, as far as the sources and receivers are concerned. Hence, the relation between t_r and t_s is the same in each. Therefore, to find out what is going on in the first scenario, we will study the second scenario (because we can figure out how this one behaves).

Consider an instantaneous inertial frame, S , of the rocket. In this frame, the rocket is momentarily at rest (at, say, $t = 0$), and then it accelerates out of the frame with acceleration g . The following discussion will be made with respect to the frame S .

Consider a series of quick light pulses emitted from the source, starting at $t = 0$. The distance the rocket has traveled out of S at time t is $gt^2/2$, so if we assume that t_s is very small, then we may say that many light pulses are emitted before the rocket moves appreciably. Likewise, the speed of the source, namely gt , is also very small. We may therefore ignore the motion of the rocket, as far as the light source is concerned.

However, the light takes a finite time to reach the receiver, and by then the receiver will be moving. We therefore *cannot* ignore the motion of the rocket when dealing with the receiver. The time it takes the light to reach the receiver is h/c , at which point the receiver has a speed of $v = g(h/c)$.² Therefore, by the usual classical Doppler effect, the time between the received pulses is³

$$t_r = \frac{t_s}{1 + (v/c)}. \quad (13.1)$$

Therefore, the frequencies, $f_r = 1/t_r$ and $f_s = 1/t_s$, are related by

$$f_r = \left(1 + \frac{v}{c}\right) f_s = \left(1 + \frac{gh}{c^2}\right) f_s. \quad (13.2)$$

Returning to the clock-on-tower scenario, we see (using the equivalence principle) that an observer on the ground will see the clock on the tower running fast, by a factor $1 + gh/c^2$. This means that the upper clock really *is* running fast, compared

²The receiver moves a tiny bit during this time, so the “ h ” here should really be replaced by a slightly smaller distance. But this yields a negligible second-order effect in the small quantity gh/c^2 , as you can show. To sum up, the displacement of the source, the speed of the source, and the displacement of the receiver are all negligible. But the speed of the receiver is quite relevant.

³Quick proof of the classical Doppler effect: As seen in frame S , when the receiver and a particular pulse meet, the next pulse is a distance ct_s behind. The receiver and this next pulse then travel toward each other at relative speed $c + v$ (as measured by someone in S). The time difference between receptions is therefore $t_r = ct_s/(c + v)$.

to the lower clock.⁴ That is,

$$\Delta t_h = \left(1 + \frac{gh}{c^2}\right) \Delta t_0. \quad (13.3)$$

A twin from Denver will be older than his twin from Boston when they meet up at a family reunion (all other things being equal, of course).

Greetings! Dear brother from Boulder,
I hear that you've gotten much older.
And please tell me why
My lower left thigh
Hasn't aged quite as much as my shoulder!

Note that the gh in eq. (13.3) is the gravitational potential energy, divided by m .

REMARK: You might object to the above derivation, because t_r is the time measured by someone in the inertial frame, S . And since the receiver is eventually moving with respect to S , we should multiply the f_r in eq. (13.2) by the usual special-relativistic time-dilation factor, $1/\sqrt{1 - (v/c)^2}$ (because the receiver's clocks are running slow relative to S , so the frequency measured by the receiver is greater than that measured in S). However, this is a second-order effect in the small quantity $v/c = gh/c^2$. We already dropped other effects of the same order, so we have no right to keep this one. Of course, if the leading effect in our final answer was second-order in v/c , then we would know that our answer was garbage. But the leading effect happens to be first order, so we can afford to be careless with the second-order effects. ♣

After a finite time has passed, the frame S will no longer be of any use to us. But we can always pick a new instantaneous rest frame of the rocket, so we can repeat the above analysis at any later time. Hence, the result in eq. (13.2) holds at all times.

This GR time-dilation effect was first measured at Harvard by Pound and Rebka in 1960. They sent gamma rays up a 20 m tower and measured the redshift (that is, the decrease in frequency) at the top. This was a notable feat indeed, considering that they were able to measure a frequency shift of gh/c^2 (which is only a few parts in 10^{15}) to within 1% accuracy.

13.3 Uniformly accelerated frame

Before reading this section, you should think carefully about the “Break or not break” problem (Problem 25) in Chapter 10. Don't look at the solution too soon, because chances are you will change your answer after a few more minutes of thought. This is a classic problem, so don't waste it by peeking!

⁴Unlike the situation where two people fly past each other (as with the usual twin paradox), we can say here that what an observer *sees* is also what actually *is*. We can say this because everyone here is in the same frame. The “turnaround” effect that was present in the twin paradox is not present now. The two clocks can be slowly moved together without anything exciting or drastic happening to their readings.

Technically, the uniformly accelerated frame we will construct has nothing to do with GR. We will not need to leave the realm of special relativity for the analysis in this section. The reason we choose to study this special-relativistic setup in detail is because it shows many similarities to genuine GR situations, such as black holes.

13.3.1 Uniformly accelerated point particle

In order to understand a uniformly accelerated frame, we first need to understand a uniformly accelerated point particle. In Section 10.7, we briefly discussed the motion of a uniformly accelerated particle, that is, one that feels a constant force in its instantaneous rest frame. Let us now take a closer look at such a particle.

Let the particle's instantaneous rest frame be S' , and let it start from rest in the inertial frame S . Let its mass be m . We know from Section 11.5.3 that the longitudinal force is the same in the two frames. Therefore, since it is constant in frame S' , it is also constant in frame S . Call it f . For convenience, let $g \equiv f/m$ (so g is the proper acceleration felt by the particle). Then in frame S we have, using the fact that f is constant,

$$f = \frac{dp}{dt} = \frac{d(m\gamma v)}{dt} \quad \Rightarrow \quad \gamma v = gt \quad \Rightarrow \quad v = \frac{gt}{\sqrt{1 + (gt)^2}}, \quad (13.4)$$

where we have set $c = 1$. As a double-check, this has the correct behavior for $t \rightarrow 0$ and $t \rightarrow \infty$. If you want to keep the c 's in, then $(gt)^2$ becomes $(gt/c)^2$, to make the units correct.

Having found the speed in frame S at time t , the position in frame S at time t is given by

$$x = \int_0^t v dt = \int_0^t \frac{gt dt}{\sqrt{1 + (gt)^2}} = \frac{1}{g} \left(\sqrt{1 + (gt)^2} - 1 \right). \quad (13.5)$$

For convenience, let P be the point (see Fig. 13.5)

$$(x_P, t_P) = (-1/g, 0). \quad (13.6)$$

Then eq. (13.5) yields

$$(x - x_P)^2 - t^2 = \frac{1}{g^2}. \quad (13.7)$$

This is the equation for a hyperbola with its center (defined as the intersection of the asymptotes) at point P . For a large acceleration g , the point P is very close to the particle's starting point. For a small acceleration, it is far away.

Everything has been fairly normal up to this point, but now the fun begins. Consider a point A on the particle's worldline at time t . From eq. (13.5), A has coordinates

$$(x_A, t_A) = \left(\frac{1}{g} \left(\sqrt{1 + (gt)^2} - 1 \right), t \right). \quad (13.8)$$

The slope of the line PA is therefore

$$\frac{t_A - t_P}{x_A - x_P} = \frac{gt}{\sqrt{1 + (gt)^2}}. \quad (13.9)$$

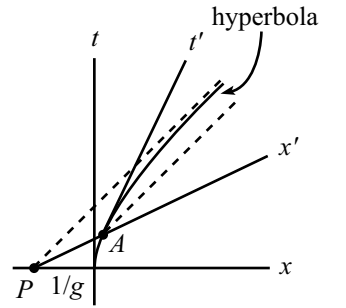


Figure 13.5

Looking at eq. (13.4), we see that this slope equals the speed of the particle at point A . But we know very well that the speed v is the slope of the particle's instantaneous x' -axis; see eq. (10.44). Therefore, the line PA and the particle's x' -axis are the same line. This holds for any arbitrary time, t . So we may say that *at any point along the particle's worldline, the line PA is the instantaneous x' -axis of the particle.* Or, said another way, *no matter where the particle is, the event at P is simultaneous with an event located at the particle, as measured in the instantaneous frame of the particle.* In other words, the particle always says that P happens “now.”⁵

Here is another strange fact. What is the distance from P to A , as measured in an instantaneous rest frame, S' , of the particle? The γ factor between frames S and S' is, using eq. (13.4), $\gamma = \sqrt{1 + (gt)^2}$. The distance between P and A in frame S is $x_A - x_P = \sqrt{1 + (gt)^2}/g$. So the distance between P and A in frame S' is (using the Lorentz transformation $\Delta x = \gamma(\Delta x' + v\Delta t')$, with $\Delta t' = 0$)

$$x'_A - x'_P = \frac{1}{\gamma}(x_A - x_P) = \frac{1}{g}. \quad (13.10)$$

This is independent of t ! Therefore, not only do we find that P is always simultaneous with the particle, in the particle's frame; we also find that *P always remains the same distance (namely $1/g$) away from the particle, as measured in the particle's instantaneous rest-frame.* This is rather strange. The particle accelerates away from point P , but it does not get further away from it (in its own frame).

REMARK: We can give a continuity argument that shows that such a point P must exist. If P is close to you, and if you accelerate away from it, then of course you get farther away from it. Everyday experience is quite valid here. But if P is sufficiently far away from you, and if you accelerate away from it, then the $at^2/2$ distance you travel away from it can easily be compensated by the decrease in distance due to length contraction (brought about by your newly acquired velocity). This effect grows with distance, so we simply need to pick P to be sufficiently far away. What this means is that every time you get out of your chair and walk to the door, there are stars very far away behind you that get closer to you as you walk away from them (as measured in your instantaneous rest frame). By continuity, then, there must exist a point P that remains the same distance from you as you accelerate away from it. ♣

13.3.2 Uniformly accelerated frame

Let's now put a collection of uniformly accelerated particles together to make a uniformly accelerated frame. The goal will be to create a frame where the distances between particles (as measured in any particle's instantaneous rest frame) remain constant.

Why is this our goal? We know from the “Break or not break” problem in Chapter 10 that if all the particles accelerate with the same proper acceleration,

⁵The point P is very much like the event horizon of a black hole. Time seems to stand still at P . And if we went more deeply into GR, we would find that time seems to stand still at the edge of a black hole, too (as viewed by someone farther away).

g , then the distances (as measured in a particle's instantaneous rest frame) grow larger. While this is a perfectly possible frame to construct, it is not desirable here for the following reason. Einstein's Equivalence Principle states that an accelerated frame is equivalent to a frame sitting on, say, the earth. We may therefore study the effects of gravity by studying an accelerated frame. But if we want this frame to look anything like the surface of the earth, we certainly can't have distances that change over time.

We therefore want to construct a *static* frame, that is, one where distances do not change (as measured in the frame). This will allow us to say that if we enclose the frame by windowless walls, then for all a person inside knows, he is standing motionless in a static gravitational field (which has a certain definite form, as we shall see).

Let's figure out how to construct the frame. We'll discuss only the acceleration of two particles here. Others can be added in an obvious manner. In the end, the desired frame as a whole is constructed by accelerating each atom in the floor of the frame with a specific proper acceleration.

From the previous subsection, we already have a particle A which is "centered" around the point P .⁶ We claim (for reasons that will become clear) that every other particle in the frame should also be "centered" around the same point P .

Consider another particle, B . Let a and b be the initial distances from P to A and B . If both particles are to be centered around P , then their proper accelerations must be, from eq. (13.6),

$$g_A = \frac{1}{a}, \quad \text{and} \quad g_B = \frac{1}{b}. \quad (13.11)$$

Therefore, in order to have all points in the frame be centered around P , we simply have to make their proper accelerations inversely proportional to their initial distances from P .

Why do we want every particle to be centered around P ? Consider two events, E_A and E_B , such that P , E_A , and E_B are collinear in Fig. 13.6. Due to construction, the line PE_AE_B is the x' -axis for both particle A and particle B , at the positions shown. From the previous subsection, we know that A is always a distance a from P , and B is always a distance b from P . Combining these facts with the fact that A and B measure their distances along the x' -axis of the same frame (at the events shown in the figure), we see that both A and B measure the distance between them to be $b - a$. This is independent of t , so A and B measure a constant distance between them. We have therefore constructed our desired static frame. This frame is often called a "Rindler space."

If a person walks around in the frame, he will think he lives in a static world where the acceleration due to gravity takes the form $g(z) \propto 1/z$, where z is the distance to a certain magical point which is located at the end of the known "universe".

What if a person releases himself from the accelerating frame, so that he forever sails through space at constant speed? He thinks he is falling, and you should

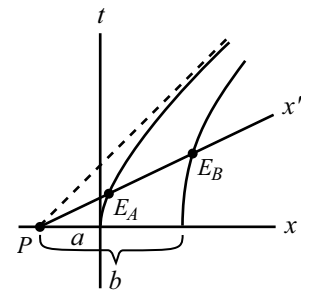


Figure 13.6

⁶This will be our shorthand notation for "traveling along a hyperbola whose center is the point P ."

convince yourself that he passes by the “magical point” in a finite proper time. But his friends who are still in the frame will see him take an infinitely long time to get to the “magical point” P . This is similar to the situation with a black hole. An outside observer will see it take an infinitely long time for a falling person to reach the “boundary” of a black hole, even though it will take a finite proper time for the person.

Our analysis shows that A and B feel a different proper acceleration, because $a \neq b$. There is no way to construct a static frame where all points feel the same proper acceleration, so it is impossible to mimic a constant gravitational field (over a finite distance) by using an accelerated frame.

13.4 Maximal-proper-time principle

The maximal-proper-time principle in General Relativity says: Given two events in spacetime, a particle under the influence of only gravity takes the path in spacetime that maximizes the proper time. For example, if you throw a ball from given coordinates (\mathbf{x}_1, t_1) , and it lands at given coordinates (\mathbf{x}_2, t_2) , then the claim is that the ball takes the path that maximizes its proper time.⁷

This is clear for a freely-moving ball in outer space, far from any massive objects. The ball travels at constant speed from one point to another, and we know that this constant-speed motion is the motion with the maximal proper time. This is true because a ball (A) moving at constant speed would see the clock on any other ball (B) slowed down due to the special-relativistic time dilation, if there were a relative speed between them. (It is assumed here that B 's non-uniform velocity is caused by a non-gravitational force acting on it.) B would therefore show a shorter elapsed time. This argument does not work the other way around, because B is not in an inertial frame and therefore cannot use the special-relativistic time-dilation result.

Consistency with Newtonian physics

The maximal-proper-time principle sounds like a plausible idea, but we already know from Chapter 5 that the path an object takes is the one that yields a stationary value of the classical action, $\int(T - V)$. We must therefore demonstrate that the “maximal”-proper-time principle reduces to the stationary-action principle, in the limit of small velocities. If this were not the case, then we'd have to throw out our theory of gravitation.

Consider a ball thrown vertically on the earth. Assume that the initial and final coordinates are fixed to be (y_1, t_1) and (y_2, t_2) . Our plan will be to assume that the maximal-proper-time principle holds, and to then show that this leads to the stationary-action principle.

⁷The principle is actually the “stationary-proper-time principle.” As with the Lagrangian formalism in Chapter 5, any type of stationary point (a maximum, minimum, or saddle point) is allowed. But although we were very careful about stating things properly in Chapter 5, we'll be a little sloppy here and just use the word “maximum,” because that's what it will generally turn out to be in the situations we will look at. However, see Problem 8.

Before being quantitative, let's get a qualitative handle on what's going on with the ball. There are two competing effects, as far as maximizing the proper time goes. On one hand, the ball wants to climb very high, because its clock will run faster there (due to the GR time dilation). But on the other hand, if it climbs very high, then it must move very fast to get there (because the total time, $t_2 - t_1$, is fixed), and this will make its clock run slow (due to the SR time dilation). So there is a tradeoff. Let's now look quantitatively at the implications of this tradeoff.

The goal is to maximize

$$\tau = \int_{t_1}^{t_2} d\tau. \quad (13.12)$$

Due to the motion of the ball, we have the usual time dilation, $d\tau = \sqrt{1 - v^2/c^2} dt$. But due to the height of the ball, we also have the gravitational time dilation, $d\tau = (1 + gy/c^2)dt$. Combining these effects gives⁸

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}} \left(1 + \frac{gy}{c^2}\right) dt. \quad (13.13)$$

Using the Taylor expansion for $\sqrt{1 - \epsilon}$, and dropping terms of order $1/c^4$ and smaller, we see that we want to maximize

$$\begin{aligned} \int_{t_1}^{t_2} d\tau &\approx \int_{t_1}^{t_2} \left(1 - \frac{v^2}{2c^2}\right) \left(1 + \frac{gy}{c^2}\right) dt \\ &\approx \int_{t_1}^{t_2} \left(1 - \frac{v^2}{2c^2} + \frac{gy}{c^2}\right) dt. \end{aligned} \quad (13.14)$$

The "1" term gives a constant, so maximizing this integral is the same as minimizing

$$mc^2 \int_{t_1}^{t_2} \left(\frac{v^2}{2c^2} - \frac{gy}{c^2}\right) dt = \int_{t_1}^{t_2} \left(\frac{mv^2}{2} - mgy\right) dt, \quad (13.15)$$

which is the classical action, as desired. For a one-dimensional gravitational problem such as this one, the action will always be a minimum, and the proper time will always be a maximum, as you can show by considering the second-order change in the action (see Exercise 10).

In retrospect, it is not surprising that this all works out. The factor of $1/2$ in the kinetic energy here comes about in exactly the same way as in the derivation in eq. (11.9), where we showed that the relativistic form of energy reduces to the familiar Newtonian expression.

13.5 Twin paradox revisited

Let's take another look at the standard twin paradox, this time from the perspective of General Relativity. We should emphasize that GR is by no means necessary for an

⁸This result is technically not correct; the two effects are intertwined in a somewhat more complicated way (see Exercise 7). But it is valid up to order v^2/c^2 , which is all we are concerned with, since we are assuming $v \ll c$.

understanding of the original formulation of the paradox (the first scenario below). We were able to solve it in Section 10.2.2, after all. The present discussion is given simply to show that the answer to an alternative formulation (the second scenario below) is consistent with what we've learned about GR. Consider the two following twin-paradox setups.

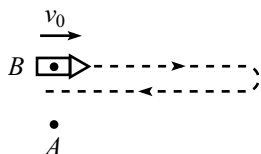


Figure 13.7

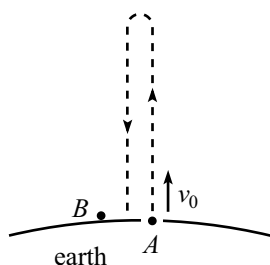


Figure 13.8

- Twin A floats freely in outer space. Twin B flies past A in a spaceship, with speed v_0 (see Fig. 13.7). At the instant they are next to each other, they both set their clocks to zero. At this same instant, B turns on the reverse thrusters of his spaceship and decelerates with proper deceleration g . B eventually reaches a farthest point from A and then accelerates back toward A , finally passing him with speed v_0 again. When they are next to each other, they compare the readings on their clocks. Which twin is younger?
- Twin B stands on the earth. Twin A is thrown upward with speed v_0 (let's say he is fired from a cannon in a hole in the ground). See Fig. 13.8. At the instant they are next to each other, they both set their clocks to zero. A rises up and then falls back down, finally passing B with speed v_0 again. When they are next to each other, they compare the readings on their clocks. Which twin is younger?

The first scenario is easily solved using special relativity. Since A is in an inertial frame, he may apply the results of special relativity. In particular, A sees B 's clock run slow, due to the usual special-relativistic time dilation. Therefore, B ends up younger at the end. Note that B cannot use the reverse reasoning, because she is not in an inertial frame.

What about the second scenario? The key point to realize is that the Equivalence Principle says that these two scenarios are *exactly the same*, as far as the twins are concerned. Twin B has no way of telling whether she is in a spaceship accelerating at g or on the surface of the earth. And A has no way of telling whether he is floating freely in outer space or in free-fall in a gravitational field.⁹ We therefore conclude that B must be younger in the second scenario, too.

At first glance, this seems incorrect, because in the second scenario, B is sitting motionless, while A is the one who is moving. It seems that B should see A 's clock running slow, due to the usual special-relativistic time dilation, and hence A should be younger. This reasoning is incorrect because it fails to take into account the gravitational time dilation. The fact of the matter is that A is higher in the gravitational field, and therefore his clock runs faster. This effect does indeed win out over the special-relativistic time dilation, and A ends up older. You can explicitly show this in Problem 11.

Note that the reasoning in this section is another way to conclude that the Equivalence Principle implies that higher clocks must run faster (in one way or

⁹This fact is made possible by the equivalence of inertial and gravitational mass. Were it not for this, different parts of A 's body would accelerate at different rates in the gravitational field in the second scenario. This would certainly clue him in to the fact that he was not floating freely in space.

another). The Equivalence Principle implies that A must be older in the second scenario, which means that there must be some height effect that makes A 's clock run fast (fast enough to win out over the special-relativistic time dilation). But it takes some more work to show that the factor is actually $1 + gh/c^2$.

Also note that the fact that A is older is consistent with the maximal-proper-time principle. In both scenarios, A is under the influence of only gravity (zero gravity in the first scenario), whereas B feels a normal force from either the spaceship's floor or the ground.

13.6 Exercises

Section 13.1: The Equivalence Principle

1. Driving on a hill

You drive up and down a hill of height h at constant speed. What should your speed be so that you age the same amount as someone standing at the base of the hill? Assume that the hill is in the shape of an isosceles triangle with altitude h .

2. Lv/c^2 and gh/c^2 *

The familiar special-relativistic “head-start” result, Lv/c^2 , looks rather similar to the gh/c^2 term in the GR time-dilation result, eq. (13.3). Imagine standing at the front of a train of length L . For small v , devise a thought experiment that explains how the Lv/c^2 result follows from the gh/c^2 result.

3. Opposite circular motion ****

A and B move at speed v ($v \ll c$) in opposite directions around a circle of radius r (so they pass each other after each half-revolution). They both see their clocks ticking at the same rate. Show this in three ways. Work in:

- The lab frame (the inertial frame whose origin is the center of the circle).
- The frame whose origin is B and whose axes remain parallel to an inertial set of axes.
- The rotating frame that is centered at the origin and rotates along with B .

Hints: See Problem 4, which is similar, although easier. And take a look at the Einstein limerick in Section 9.2.

Section 13.2: Uniformly accelerated frame

4. Using rapidity *

Another way to derive the v in eq. (13.4) is to use the $v = \tanh(g\tau)$ rapidity result (where τ is the particle’s proper time) from Section 10.7. Use time dilation to show that this implies $gt = \sinh(g\tau)$, and hence eq. (13.4).

5. Various quantities *

A particle starts at rest and accelerates with proper acceleration g . Let τ be the time on the particle’s clock. Using the v from eq. (13.4), use time dilation to show that the time t in the original inertial frame, the speed of the particle, and the associated γ factor are given by (with $c = 1$)

$$gt = \sinh(g\tau), \quad v = \tanh(g\tau), \quad \gamma = \cosh(g\tau). \quad (13.16)$$

6. Redshift **

We found in Section 13.2 that a clock at the back of a rocket will see a clock at the front run fast by a factor $1 + gh/c^2$. However, we ignored higher-order effects in $1/c^2$, so for all we know, the factor is actually, say, e^{gh/c^2} , or perhaps $\ln(gh/c^2)$, and we found only the first term in the Taylor series.

- (a) For the uniformly accelerated frame in Section 13.3.2, show that the factor is in fact exactly $1 + g_b h/c^2$, where g_b is the acceleration of the back of the rocket. Show this by lining up a series of clocks and looking at the successive factors between them.
- (b) By the same reasoning, it follows that the front clock sees the back clock running slow by a factor $1 - g_f h/c^2$, where g_f is the acceleration of the front. Show explicitly that $(1 + g_b/c^2)(1 - g_f h/c^2) = 1$, as must be the case, because a clock can't gain time with respect to itself.

7. Gravity and speed combined **

Use a Minkowski diagram to do this problem (in the spirit of Problem 10.24, "Acceleration and redshift").

A rocket accelerates with proper acceleration g toward a planet. As measured in the instantaneous inertial frame of the rocket, the planet is a distance x away and moves at speed v . Everything is in one dimension here.

As measured in the *accelerating* frame of the rocket, show that the planet's clock runs at a rate (with $c = 1$),

$$dt_p = dt_r(1 + gx)\sqrt{1 - v^2}. \quad (13.17)$$

And show that the planet's speed is

$$V = (1 + gx)v. \quad (13.18)$$

Note that if we combine these two results to eliminate v , and if we then invoke the equivalence principle, we arrive at the result that a clock moving at height h and speed V in a gravitational field is seen by someone on the ground to run at a rate (putting the c 's back in),

$$\sqrt{\left(1 + \frac{gh}{c^2}\right)^2 - \frac{V^2}{c^2}}. \quad (13.19)$$

8. Speed in accelerating frame *

In the setup in Problem 6, use eq. (13.20) to find the speed of the planet, $dx/d\tau$, as a function of τ . What is the maximum value of this speed, in terms of g and the initial distance, L ?

9. Accelerating stick's length **

Consider a uniformly accelerated frame consisting of a stick, the ends of which have worldlines given by the curves in Fig. 13.6 (so the stick has proper length $b - a$). At time t in the lab frame, we know that a point that undergoes acceleration g has position $\sqrt{1 + (gt)^2}/g$ relative to the point P in Fig. 13.6.

An observer in the original inertial frame will see the stick being length-contracted by different factors along its length, because different points move with different speeds (at a given time in the original frame). Show, by doing the appropriate integral, that this observer will conclude that the stick always has proper length $b - a$.

Section 13.3: Maximal proper-time principle

10. Maximum proper time *

Show that the extremum of the gravitational action in eq. (13.15) is always a minimum. Do this considering a function, $y(t) = y_0(t) + \xi(t)$, where y_0 is the path that extremizes the action, and ξ is a small variation.

Section 13.4: Twin paradox revisited

11. Symmetric twin non-paradox **

Two twins travel in opposite directions at speed v ($v \ll c$) with respect to the earth. They synchronize their clocks when they pass each other. They travel to stars located a distance ℓ away, and then decelerate and accelerate back up to speed v in the opposite direction (uniformly, and in a short time compared to the total journey time).

In the frame of the earth, it is obvious (from symmetry) that both twins age the same amount by the time they pass each other again. Reproduce this result by working in the frame of one of the twins.

13.7 Problems

Section 13.1: The Equivalence Principle

1. Airplane's speed

A plane flies at constant height h . What should its speed be so that an observer on the ground sees the plane's clock tick at the same rate as a ground clock? (Assume $v \ll c$.)

2. Clock on tower **

A clock starts on the ground and then moves up a tower at constant speed v . It sits on top of the tower for a time T and then descends at constant speed v . If the tower has height h , how long should the clock sit at the top so that it comes back showing the same time as a clock that remained on the ground? (Assume $v \ll c$.)

3. Circular motion **

Person B moves at speed v (with $v \ll c$) in a circle of radius r around person A . By what fraction does B 's clock run slower than A 's? Calculate this in three ways. Work in:

- A 's frame.
- The frame whose origin is B and whose axes remain parallel to an inertial set of axes.
- The rotating frame that is centered at A and rotates around A with the same frequency as B .

4. More circular motion **

A and B move at speed v ($v \ll c$) in a circle of radius r , at diametrically opposite points. They both see their clocks ticking at the same rate. Show this in three ways. Work in:

- The lab frame (the inertial frame whose origin is the center of the circle).
- The frame whose origin is B and whose axes remain parallel to an inertial set of axes.
- The rotating frame that is centered at the origin and rotates with the same frequency as A and B .

Section 13.2: Uniformly accelerated frame

5. Getting way ahead ****

A rocket with proper length L accelerates from rest, with proper acceleration g (where $gL \ll c^2$). Clocks are located at the front and back of the rocket. If we look at this setup in the frame of the rocket, then the general-relativistic time-dilation effect tells us that the times on the two clocks are related by

$t_f = (1 + gL/c^2)t_b$. Therefore, if we look at things in the ground frame, then the times on the two clocks are related by

$$t_f = t_b \left(1 + \frac{gL}{c^2} \right) - \frac{Lv}{c^2},$$

where the last term is the standard special-relativistic “head-start” result. Derive the above relation by working entirely in the ground frame.¹⁰

6. Accelerator’s point of view ***

A rocket starts at rest relative to a planet, a distance ℓ away. It accelerates toward the planet with proper acceleration g . Let τ and t be the readings on the rocket’s and planet’s clocks, respectively.

- (a) Show that when the astronaut’s clock reads τ , he observes the rocket-planet distance, x (as measured in his instantaneous inertial frame), to be given by

$$1 + gx = \frac{1 + g\ell}{\cosh(g\tau)}. \quad (13.20)$$

- (b) Show that when the astronaut’s clock reads τ , he observes the time, t , on the planet’s clock to be given by

$$gt = (1 + g\ell) \tanh(g\tau). \quad (13.21)$$

The results from Exercises 5 and 7 will be useful here.

7. Lv/c^2 revisited **

You stand at rest relative to a rocket that has synchronized clocks at its ends. It is then arranged for you and the rocket to move with relative speed v . A reasonable question to now ask is: As viewed by you, what is the difference in readings on the clocks located at the ends of the rocket?

It turns out that this question cannot be answered without further information on how you and the rocket got to be moving with relative speed v . There are two basic ways this relative speed can come about. The rocket can accelerate while you sit there, or you can accelerate while the rocket sits there. Using the results from Problems 5 and 6, explain what the answers to the above question are in these two cases.

8. Circling the earth **

Clock A sits at rest on the earth, and clock B circles the earth in an orbit that skims along the ground. Both A and B are essentially at the same radius, so

¹⁰You may find this relation surprising, because it implies that the front clock will eventually be an arbitrarily large time ahead of the back clock, in the ground frame. (The subtractive Lv/c^2 term is bounded by L/c and will therefore eventually become negligible compared to the additive, and unbounded, $(gL/c^2)t_b$ term.) But both clocks are doing basically the same thing relative to the ground frame, so how can they eventually differ by so much? Your job is to find out.

the GR time-dilation effect yields no difference in their times. But B is moving relative to A , so A will see B running slow, due to the usual SR time-dilation effect. The orbiting clock, B , will therefore show a *smaller* elapsed proper time each time it passes A . In other words, the clock under the influence of only gravity (B) does *not* show the maximal proper time, in conflict with what we have been calling the maximal-proper-time principle. Explain.

Section 13.4: Twin paradox revisited

9. Twin paradox *

A spaceship travels at speed v ($v \ll c$) to a distant star. Upon reaching the star, it decelerates and then accelerates back up to speed v in the opposite direction (uniformly, and in a short time compared to the total journey time). By what fraction does the traveler age less than her twin on earth? (Ignore the gravity from the earth.) Work in:

- (a) The earth frame.
- (b) The spaceship frame.

10. Twin paradox again **

- (a) Answer the previous problem, except now let the spaceship turn around by moving in a small semicircle while maintaining speed v .
- (b) Answer the previous problem, except now let the spaceship turn around by moving in an arbitrary manner. The only constraints are that the turn-around is done quickly (compared to the total journey time), and that it is contained in a small region of space (compared to the earth-star distance).

11. Twin paradox times ***

- (a) In the first scenario in Section 13.5, calculate the ratio of B 's elapsed time to A 's, in terms of v_0 and g . Assume that $v_0 \ll c$, and drop high-order terms.
- (b) Do the same for the second scenario in Section 13.5. Do this from scratch using the time dilations, and then check that your answer agrees (within the accuracy of the calculations) with part (a), as the equivalence principle demands.

13.8 Solutions

1. Airplane's speed

An observer on the ground sees the plane's clock run slow by a factor $\sqrt{1 - v^2/c^2}$ due to SR time dilation. But he also sees it run fast by a factor $(1 + gh/c^2)$ due to GR time dilation. We therefore want the product of these two factors to equal 1. Using the standard Taylor-series approximation for slow speeds in the first factor, we find

$$\left(1 - \frac{v^2}{2c^2}\right) \left(1 + \frac{gh}{c^2}\right) = 1 \quad \implies \quad 1 - \frac{v^2}{2c^2} + \frac{gh}{c^2} - \mathcal{O}\left(\frac{1}{c^4}\right) = 1. \quad (13.22)$$

Neglecting the small $1/c^4$ term, and cancelling the 1's, yields $v = \sqrt{2gh}$.

Interestingly, $\sqrt{2gh}$ is also the answer to a standard question from Newtonian physics, namely, how fast must you throw a ball straight up if you want it to reach a height h ?

2. Clock on tower

The SR time-dilation factor is $\sqrt{1 - v^2/c^2} \approx 1 - v^2/2c^2$. The clock therefore loses a fraction $v^2/2c^2$ of the time elapsed during its motion up and down the tower. The upward journey takes a time h/v , and likewise for the downward trip, so the time loss due to the SR effect is

$$\left(\frac{v^2}{2c^2}\right) \left(\frac{2h}{v}\right) = \frac{vh}{c^2}. \quad (13.23)$$

Our goal is to balance this time loss with the time gain due to the GR time-dilation effect. If the clock sits on top of the tower for a time T , then the time gain is

$$\left(\frac{gh}{c^2}\right) T. \quad (13.24)$$

But we must not forget also the increase in time due to the height gained while the clock is in motion. During its motion, the clock's average height is $h/2$. The total time in motion is $2h/v$, so the GR time gain while the clock is moving is

$$\left(\frac{g(h/2)}{c^2}\right) \left(\frac{2h}{v}\right) = \frac{gh^2}{c^2v}. \quad (13.25)$$

Setting the total change in the clock's time equal to zero gives

$$-\frac{vh}{c^2} + \frac{gh}{c^2}T + \frac{gh^2}{c^2v} = 0 \quad \implies \quad -v + gT + \frac{gh}{v} = 0. \quad (13.26)$$

Therefore,

$$T = \frac{v}{g} - \frac{h}{v}. \quad (13.27)$$

REMARKS: Note that we must have $v > \sqrt{gh}$ in order for a positive solution for T to exist. If $v < \sqrt{gh}$, then the SR effect is too small to cancel out the GR effect, even if the clock spends no time sitting at the top. If $v = \sqrt{gh}$, then $T = 0$, and we essentially have the same situation as in Exercise 1. Note also that if v is very large compared to \sqrt{gh} (but still small compared to c , so that our $\sqrt{1 - v^2/c^2} \approx 1 - v^2/2c^2$ approximation is valid), then $T \approx v/g$, which is independent of h . ♣

3. Circular motion

- (a) In A 's frame, there is only the SR time-dilation effect. A sees B move at speed v , so B 's clock runs slow by a factor of $\sqrt{1 - v^2/c^2}$. And since $v \ll c$, we may use the Taylor series to approximate this as $1 - v^2/2c^2$.
- (b) In this frame, there are both SR and GR time-dilation effects. A moves at speed v with respect to B in this frame, so there is the SR effect that A 's clock runs slow by a factor $\sqrt{1 - v^2/c^2} \approx 1 - v^2/2c^2$.
But B undergoes an acceleration of $a = v^2/r$ toward A , so there is also the GR effect that A 's clock runs fast by a factor $1 + ar/c^2 = 1 + v^2/c^2$.
Multiplying these two effects together, we find (to lowest order) that A 's clock runs fast by a factor $1 + v^2/2c^2$. This means (to lowest order) that B 's clock runs slow by a factor $1 - v^2/2c^2$, in agreement with the answer to part (a).
- (c) In this frame, there is no relative motion between A and B , so there is only the GR time-dilation effect. The gravitational field (that is, the centripetal acceleration) at a distance x from the center is $g_x = x\omega^2$. Imagine lining up a series of clocks along a radius, with separation dx . Then the GR time-dilation result tells us that each clock loses a fraction $g_x dx/c^2 = x\omega^2 dx/c^2$ of time relative to the clock just inside it. Integrating these fractions from $x = 0$ to $x = r$ shows that B 's clock loses a fraction $r^2\omega^2/2c^2 = v^2/2c^2$, compared to A 's clock. This agrees with the results in parts (a) and (b).

4. More circular motion

- (a) In the lab frame, the situation is symmetric with respect to A and B . Therefore, if A and B are decelerated in a symmetric manner and brought together, then their clocks must read the same time.

Assume (in the interest of obtaining a contradiction), that A sees B 's clock run slow. Then after an arbitrarily long time, A will see B 's clock an arbitrarily large time behind his. Now bring A and B to a stop. There is no possible way that the stopping motion can make B 's clock gain an arbitrarily large amount of time, as seen by A . This is true because everything takes place in a finite region of space, so there is an upper bound on the GR time-dilation effect (because it behaves like gh/c^2 , and h is bounded). Therefore, A will end up seeing B 's clock reading less. This contradicts the result of the previous paragraph.

REMARK: Note how this problem differs from the problem where A and B move with equal speeds directly away from each other, and then reverse directions and head back to meet up again.

For this new "linear" problem, the symmetry reasoning in the first paragraph above still holds; they will indeed have the same clock readings when they meet up again. But the reasoning in the second paragraph does not hold (it better not, because each person does *not* see the other person's clock running at the same rate). The error is that in this linear scenario, the experiment is not contained in a small region of space, so the turning-around effects of order gh/c^2 become arbitrarily large as the time of travel becomes arbitrarily large, since h grows with time (see Problem 9). ♣

- (b) In this frame, there are both SR and GR time-dilation effects. A moves at speed $2v$ with respect to B in this frame (we don't need to use the relativistic velocity-addition formula, because $v \ll c$), so this gives the SR effect that A 's clock runs slow by a factor $\sqrt{1 - (2v)^2/c^2} \approx 1 - 2v^2/c^2$.

But B undergoes an acceleration of $a = v^2/r$ toward A , so there is also the GR effect that A 's clock runs fast by a factor $1 + a(2r)/c^2 = 1 + 2v^2/c^2$ (because they are separated by a distance $2r$).

Multiplying these two effects together, we find (to lowest order) that the two clocks run at the same rate.

- (c) In this frame, there is no relative motion between A and B . Hence, there is only the GR effect. But A and B are both at the same gravitational potential, because they are at the same radius. Therefore, they both see the clocks running at the same rate.

If you want, you can line up a series of clocks along the diameter between A and B , as we did along a radius in part (c) of Problem 3. The clocks will gain time as you march in toward the center, and then lose back the same amount of time as you march back out to the diametrically opposite point.

5. Getting way ahead

The explanation of why the two clocks show different times in the ground frame is the following. The rocket becomes increasingly length contracted in the ground frame, which means that the front end isn't traveling as fast as the back end. Therefore, the time-dilation factor for the front clock isn't as large as that for the back clock. So the front clock loses less time relative to the ground, and hence ends up ahead of the back clock. Of course, it's not at all obvious that everything works out quantitatively, and that the front clock eventually ends up an arbitrarily large time ahead of the back clock. In fact, it's quite surprising that this is the case, because the above difference in speeds is rather small. But let's now show that the above explanation does indeed account for the difference in the clock readings.

Let the back of the rocket be located at position x . Then the front is located at position $x + L\sqrt{1-v^2}$ (with $c = 1$), due to the length contraction. Taking the time derivatives of the two positions, we see that the speeds of the back and front are (with $v \equiv dx/dt$)¹¹

$$v_b = v, \quad \text{and} \quad v_f = v(1 - L\gamma\dot{v}). \quad (13.28)$$

For v_b , we will simply invoke the result in eq. (13.4),

$$v_b = v = \frac{gt}{\sqrt{1 + (gt)^2}}, \quad (13.29)$$

where t is the time in the ground frame.

Having found v , we must now find the γ -factors associated with the speeds of the front and back of the rocket. The γ -factor associated with the speed of the back (namely v) is

$$\gamma_b = \frac{1}{\sqrt{1-v^2}} = \sqrt{1 + (gt)^2}. \quad (13.30)$$

The γ -factor associated with the speed of the front, $v_f = v(1 - L\gamma\dot{v})$, is a little harder to obtain. We must first calculate \dot{v} . From eq. (13.29), we find $\dot{v} = g/(1 + g^2t^2)^{3/2}$, which gives

$$v_f = v(1 - L\gamma\dot{v}) = \frac{gt}{\sqrt{1 + (gt)^2}} \left(1 - \frac{gL}{1 + g^2t^2} \right). \quad (13.31)$$

¹¹Since these speeds are not equal, there is of course an ambiguity concerning which speed we should use in the length-contraction factor, $\sqrt{1-v^2}$. Equivalently, the rocket actually doesn't have one inertial frame that describes all of it. But you can show that any differences arising from this ambiguity are of higher order in gL/c^2 than we need to be concerned with.

The γ -factor (or rather $1/\gamma$, which is what we'll be concerned with) associated with this speed can now be found as follows. In the first line below, we ignore the higher-order $(gL)^2$ term, because it is really $(gL/c^2)^2$, and we are assuming that gL/c^2 is small. And in obtaining the third line, we use the Taylor-series approximation, $\sqrt{1-\epsilon} \approx 1-\epsilon/2$.

$$\begin{aligned} \frac{1}{\gamma_f} = \sqrt{1-v_f^2} &\approx \sqrt{1 - \frac{g^2 t^2}{1+g^2 t^2} \left(1 - \frac{2gL}{1+g^2 t^2}\right)} \\ &= \frac{1}{\sqrt{1+g^2 t^2}} \sqrt{1 + \frac{2g^3 t^2 L}{1+g^2 t^2}} \\ &\approx \frac{1}{\sqrt{1+g^2 t^2}} \left(1 + \frac{g^3 t^2 L}{1+g^2 t^2}\right). \end{aligned} \quad (13.32)$$

We can now calculate the time that each clock shows, at time t in the ground frame. The time on the back clock changes according to $dt_b = dt/\gamma_b$, so eq. (13.30) gives

$$t_b = \int_0^t \frac{dt}{\sqrt{1+g^2 t^2}}. \quad (13.33)$$

The integral¹² of $1/\sqrt{1+x^2}$ is $\sinh^{-1} x$. Letting $x \equiv gt$, this gives

$$gt_b = \sinh^{-1}(gt). \quad (13.34)$$

The time on the front clock changes according to $dt_f = dt/\gamma_f$, so eq. (13.32) gives

$$t_f = \int_0^t \frac{dt}{\sqrt{1+g^2 t^2}} + \int_0^t \frac{g^3 t^2 L dt}{(1+g^2 t^2)^{3/2}}. \quad (13.35)$$

The integral¹³ of $x^2/(1+x^2)^{3/2}$ is $\sinh^{-1} x - x/\sqrt{1+x^2}$. Letting $x \equiv gt$, this gives

$$gt_f = \sinh^{-1}(gt) + (gL) \left(\sinh^{-1}(gt) - \frac{gt}{\sqrt{1+g^2 t^2}} \right). \quad (13.36)$$

Using eqs. (13.29), (13.34), and (13.36), we may rewrite this as

$$gt_f = gt_b(1+gL) - gLv. \quad (13.37)$$

Dividing by g , and putting the c 's back in to make the units correct, we finally have

$$t_f = t_b \left(1 + \frac{gL}{c^2} \right) - \frac{Lv}{c^2}, \quad (13.38)$$

as we wanted to show.

REMARK: Looked at from the reverse point of view, this calculation, which uses only special-relativity concepts, demonstrates that someone at the back of a rocket sees a clock at the front running fast by a factor $(1+gL/c^2)$. There are, however, far easier ways of deriving this, as we saw in Section 13.2 and in Problem 10.24 ("Acceleration and redshift"). ♣

¹²To derive this, make the substitution $x \equiv \sinh \theta$.

¹³Again, to derive this, make the substitution $x \equiv \sinh \theta$.

6. Accelerator's point of view

- (a) **First Solution:** Eq. (13.5) says that the distance traveled by the rocket (as measured in the original inertial frame), as a function of the time in the inertial frame, is

$$d = \frac{1}{g} \left(\sqrt{1 + (gt)^2} - 1 \right). \quad (13.39)$$

An inertial observer on the planet therefore measures the rocket-planet distance to be

$$x = \ell - \frac{1}{g} \left(\sqrt{1 + (gt)^2} - 1 \right). \quad (13.40)$$

The rocket observer will see this length being contracted by a factor γ . Using the result of Exercise 5, we have $\gamma = \sqrt{1 + (gt)^2} = \cosh(g\tau)$. So the rocket-planet distance, as measured in the instantaneous inertial frame of the rocket, is

$$x = \frac{\ell - \frac{1}{g} (\cosh(g\tau) - 1)}{\cosh(g\tau)} \quad \Longrightarrow \quad 1 + gx = \frac{1 + g\ell}{\cosh(g\tau)}, \quad (13.41)$$

as desired.

Second Solution: Eq. (13.18) gives the speed of the planet in the accelerating frame of the rocket. Using the results of Exercise 5 to write v in terms of τ , we have (with $c = 1$)

$$\frac{dx}{d\tau} = -(1 + gx) \tanh(g\tau). \quad (13.42)$$

Separating variables and integrating gives

$$\begin{aligned} \int \frac{dx}{1 + gx} &= - \int \tanh(g\tau) d\tau &\Longrightarrow & \ln(1 + gx) = - \ln(\cosh(g\tau)) + C \\ & &\Longrightarrow & 1 + gx = \frac{A}{\cosh(g\tau)}. \end{aligned} \quad (13.43)$$

Since the initial condition is $x = \ell$ when $\tau = 0$, we must have $A = 1 + g\ell$, which gives eq. (13.20), as desired.

- (b) Eq. (13.17) says that the planet's clock runs fast (or slow) according to

$$dt = d\tau (1 + gx) \sqrt{1 - v^2}. \quad (13.44)$$

The results of Exercise 5 yield $\sqrt{1 - v^2} = 1/\cosh(g\tau)$. Combining this with the result for $1 + gx$ above, and integrating, gives

$$\int dt = \int \frac{(1 + g\ell) d\tau}{\cosh^2(g\tau)} \quad \Longrightarrow \quad gt = (1 + g\ell) \tanh(g\tau), \quad (13.45)$$

as desired.

7. Lv/c^2 revisited

Consider first the case where the rocket accelerates while you sit there. Problem 5 is exactly relevant here, and it tells us that in your frame the clock readings are related by

$$t_f = t_b \left(1 + \frac{gL}{c^2} \right) - \frac{Lv}{c^2}. \quad (13.46)$$

You will eventually see the front clock an arbitrarily large time ahead of the back clock. Note that for small times (before things become relativistic), the standard Newtonian result, $v \approx gt_b$, is valid, so we have

$$t_f \approx \left(t_b + \frac{Lv}{c^2} \right) - \frac{Lv}{c^2} = t_b. \quad (13.47)$$

We see that in the setup where the rocket is the one that accelerates, both clocks show essentially the same time near the start (the leading term in the time difference is of order $(v/c)^2$). This makes sense; both clocks have essentially the same speed at the beginning, so to lowest order their γ factors are the same, so the clocks run at the same rate. But eventually the front clock will get ahead of the back clock.

Now consider the case where you accelerate while the rocket sits there. Problem 6 is relevant here, if we let the rocket in that problem now become you, and if we let two planets a distance L apart become the two ends of the rocket. The times you observe on the front and back clocks on the rocket are then, using eq. (13.45) and assuming that you are accelerating toward the rocket,

$$gt_f = (1 + g\ell) \tanh(g\tau), \quad \text{and} \quad gt_b = (1 + g(\ell + L)) \tanh(g\tau). \quad (13.48)$$

But from Exercise 5, we know that your speed relative to the rocket is $v = \tanh g\tau$. Eq. (13.48) therefore gives $t_b = t_f + Lv$, or $t_b = t_f + Lv/c^2$ with the c^2 . So in this case we arrive at the standard Lv/c^2 “head-start” result.

The point here is that in this second case, the clocks are synchronized in the rocket frame, and this is the assumption that went into our derivation of the Lv/c^2 result in Chapter 10. In the first case above where the rocket accelerates, the clocks are *not* synchronized in the rocket frame (except right at the start), so it’s not surprising that we don’t obtain the Lv/c^2 result.

8. Circling the earth

This is one setup where we really need to use the correct term, *stationary*-proper-time principle. It turns out that B ’s path yields a saddle point for the proper time. The value at this saddle point is less than A ’s proper time, but this is irrelevant, because we only care about local extrema, not global ones.

B ’s path is a saddle point because there exist nearby paths that give both a larger and smaller proper time.¹⁴ The proper time can be made smaller by having B speed up and slow down. This will cause a net increase in the time-dilation effect as viewed by A , thereby yielding a smaller proper time.¹⁵ The proper time can be made larger by having B take a nearby path that doesn’t quite form a great circle on the earth. (Imagine the curve traced out by a rubber band that has just begun to slip away from a great-circle position.) This path is shorter, so B won’t have to travel as fast to get back in a given time, so the time-dilation effect will be smaller as viewed by A , thereby yielding a larger proper time.

¹⁴The differences are in fact second-order ones, because the first-order ones vanish due to the fact that the path satisfies the Euler-Lagrange equations for the Lagrangian in eq. (13.15).

¹⁵This is true for the same reason that a person who travels at constant speed in a straight line between two points will show a larger proper time than a second person who speeds up and slows down. This follows directly from SR time dilation, as viewed by the first person. If you want, you can imagine unrolling B ’s circular orbit into a straight line, and then invoke the result just mentioned. As far as SR time-dilation effects from clock A ’s point of view go, it doesn’t matter if the circle is unrolled into a straight line.

9. Twin paradox

- (a) In the earth frame, the spaceship travels at speed v for essentially the whole time. Therefore, the traveler ages less by a fraction $\sqrt{1 - v^2/c^2} \approx 1 - v^2/2c^2$. The fractional loss of time is thus $v^2/2c^2$. The time-dilation effect will be different during the short turning-around period, but this is negligible.
- (b) Let the distance to the star be ℓ , as measured in the frame of the earth (but the difference in lengths in the two frames is negligible in this problem), and let the turnaround take a time T . Then the given information says that $T \ll (2\ell)/v$.

During the constant-speed part of the trip, the traveler sees the earth clock running slow by a fraction $\sqrt{1 - v^2/c^2} \approx 1 - v^2/2c^2$. The time for this constant-speed part is $2\ell/v$, so the earth clock loses a time of $(v^2/2c^2)(2\ell/v) = v\ell/c^2$.

However, during the turnaround time, the spaceship is accelerating toward the earth, so the traveler sees the earth clock running fast, due to the GR time dilation. The magnitude of the acceleration is $a = 2v/T$, because the spaceship goes from velocity v to $-v$ in time T . The earth clock therefore runs fast by a factor $1 + a\ell/c^2 = 1 + 2v\ell/Tc^2$. This happens for a time T , so the earth clock gains a time of $(2v\ell/Tc^2)T = 2v\ell/c^2$.

Combining the results of the previous two paragraphs, we see that the earth clock gains a time of $2v\ell/c^2 - v\ell/c^2 = v\ell/c^2$. This is a fraction $(v\ell/c^2)/(2\ell/v) = v^2/2c^2$ of the total time, in agreement with part (a).

10. Twin paradox again

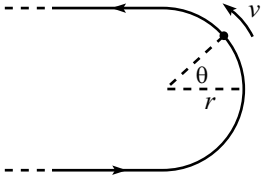


Figure 13.9

- (a) The only difference between this problem and the previous one is the nature of the turnaround, so all we need to show here is that the traveler still sees the earth clock gain a time of $2v\ell/c^2$ during the turnaround.

Let the radius of the semicircle be r . Then the magnitude of the acceleration is $a = v^2/r$. Let θ be the angle shown in Fig. 13.9. For a given θ , the earth is at a height of essentially $\ell \cos \theta$ in the gravitational field felt by the spaceship. The fractional time that the earth gains while the traveler is at angle θ is therefore $ah/c^2 = (v^2/r)(\ell \cos \theta)/c^2$. Integrating this over the time of the turnaround, and using $dt = r d\theta/v$, we see that the earth gains a time of

$$\Delta t = \int_{-\pi/2}^{\pi/2} \left(\frac{v^2 \ell \cos \theta}{rc^2} \right) \left(\frac{r d\theta}{v} \right) = \frac{2v\ell}{c^2}, \quad (13.49)$$

during the turnaround, as we wanted to show.

- (b) Let the acceleration vector at a given instant be \mathbf{a} , and let $\boldsymbol{\ell}$ be the vector from the spaceship to the earth. Note that since the turnaround is done in a small region of space, $\boldsymbol{\ell}$ is essentially constant here.

The earth is at a height of essentially $\hat{\mathbf{a}} \cdot \boldsymbol{\ell}$ in the gravitational field felt by the spaceship. (The dot product just gives the cosine term in the above solution to part (a).) The fractional time gain, ah/c^2 , is therefore equal to $|\mathbf{a}|(\hat{\mathbf{a}} \cdot \boldsymbol{\ell})/c^2 = \mathbf{a} \cdot \boldsymbol{\ell}/c^2$. Integrating this over the time of the turnaround, we see that the earth gains a time of

$$\begin{aligned} \Delta t &= \int_{t_i}^{t_f} \frac{\mathbf{a} \cdot \boldsymbol{\ell}}{c^2} dt = \frac{\boldsymbol{\ell}}{c^2} \cdot \int_{t_i}^{t_f} \mathbf{a} dt \\ &= \frac{\boldsymbol{\ell}}{c^2} \cdot (\mathbf{v}_f - \mathbf{v}_i) \end{aligned}$$

$$\begin{aligned}
&= \frac{\ell \cdot (2\mathbf{v}_f)}{c^2} \\
&= \frac{2v\ell}{c^2}, \tag{13.50}
\end{aligned}$$

during the turnaround, as we wanted to show. The whole point here is that no matter what complicated motion the traveler undergoes during the turnaround, the total effect is to simply change the velocity from \mathbf{v} outward to \mathbf{v} inward.

11. Twin paradox times

(a) As viewed by A , the times of the twins are related by

$$dt_B = \sqrt{1 - v^2} dt_A. \tag{13.51}$$

Assuming $v_0 \ll c$, we may say that $v(t_A)$ is essentially equal to $v_0 - gt_A$, so the out and back parts of the trip each take a time of essentially v_0/g in A 's frame. The total elapsed time on B 's clock is therefore

$$\begin{aligned}
T_B = \int dt_B &\approx 2 \int_0^{v_0/g} \sqrt{1 - v^2} dt_A \\
&\approx 2 \int_0^{v_0/g} \left(1 - \frac{v^2}{2}\right) dt_A \\
&\approx 2 \int_0^{v_0/g} \left(1 - \frac{1}{2}(v_0 - gt)^2\right) dt \\
&= 2 \left(t + \frac{1}{6g}(v_0 - gt)^3 \right) \Big|_0^{v_0/g} \\
&= \frac{2v_0}{g} - \frac{v_0^3}{3gc^2}, \tag{13.52}
\end{aligned}$$

where we have put the c 's back in to make the units right. The ratio of B 's elapsed time to A 's is therefore

$$\frac{T_B}{T_A} \approx \frac{T_B}{2v_0/g} \approx 1 - \frac{v_0^2}{6c^2}. \tag{13.53}$$

(b) As viewed by B , the relation between the twins' times is given by eq. (13.13),

$$dt_A = \sqrt{1 - \frac{v^2}{c^2}} \left(1 + \frac{gy}{c^2}\right) dt_B. \tag{13.54}$$

Assuming $v_0 \ll c$, we may say that $v(t_B)$ is essentially equal to $v_0 - gt_B$, and A 's height is essentially equal to $v_0 t_B - gt_B^2/2$. The up and down parts of the trip each take a time of essentially v_0/g in B 's frame. Therefore, the total elapsed time on A 's clock is (using the approximation in eq. (13.14), and dropping the c 's)

$$\begin{aligned}
T_A = \int dt_A &\approx 2 \int_0^{v_0/g} \left(1 - \frac{v^2}{2} + gy\right) dt_B \\
&\approx 2 \int_0^{v_0/g} \left(1 - \frac{1}{2}(v_0 - gt)^2 + g(v_0 t - gt^2/2)\right) dt.
\end{aligned}$$

$$\begin{aligned}
&= 2 \left(t + \frac{1}{6g}(v_0 - gt)^3 + g \left(\frac{v_0 t^2}{2} - \frac{gt^3}{6} \right) \right) \Big|_0^{v_0/g} \\
&= \frac{2v_0}{g} - \frac{v_0^3}{3g} + g \left(\frac{v_0^3}{g^2} - \frac{v_0^3}{3g^2} \right) \\
&= \frac{2v_0}{g} + \frac{v_0^3}{3gc^2}, \tag{13.55}
\end{aligned}$$

where we have put the c 's back in to make the units right. We therefore have

$$\frac{T_A}{T_B} \approx \frac{T_A}{2v_0/g} \approx 1 + \frac{v_0^2}{6c^2} \quad \Longrightarrow \quad \frac{T_B}{T_A} \approx 1 - \frac{v_0^2}{6c^2}, \tag{13.56}$$

up to corrections of order v_0^4/c^4 . This agrees with the result found in part (a), as the equivalence principle requires.