Chapter 9

Accelerated Frames of Reference

Newton's laws hold only in inertial frames of reference. However, there are many non-inertial (that is, accelerated) frames of reference that we might reasonably want to study, such as elevators, merry-go-rounds, and so on. Is there any possible way to modify Newton's laws so that they hold in non-inertial frames, or do we have to give up entirely on $\mathbf{F} = m\mathbf{a}$?

It turns out that we can indeed hold onto our good friend $\mathbf{F} = m\mathbf{a}$, provided that we introduce some new "fictitious" forces. These are forces that a person in the accelerated frame thinks exist. If she applies $\mathbf{F} = m\mathbf{a}$ while including these new forces, then she will get the correct answer for the acceleration, $\mathbf{a}$, as measured with respect to her frame.

To be quantitative about all this, we'll have to spend some time determining how the coordinates (and their derivatives) in an accelerated frame relate to those in an inertial frame. But before diving into that, let's look at a simple example which demonstrates the basic idea of fictitious forces.

Example (The train): Imagine that you are standing on a train that is accelerating to the right with acceleration a. If you wish to remain at the same spot on the train, there must be a friction force between the floor and your feet, with magnitude $F_{\text{f}} = ma$, pointing to the right. Someone standing in the inertial frame of the ground will simply interpret the situation as, "The friction force, $F_{\text{f}} = ma$, causes your acceleration, a."

How do you interpret the situation, in the frame of the train? (Imagine that there are no windows, so that all you see is the inside of the train.) As we will show below in eq. (9.11), you will feel a fictitious *translation* force, $F_{\text{trans}} = -ma$, pointing to the left. You will therefore interpret the situation as, "In my frame (the frame of the train), the friction force $F_{\text{f}} = ma$ pointing to my right exactly cancels the mysterious $F_{\text{trans}} = -ma$ force pointing to my left, resulting in zero acceleration (in my frame)."

Of course, if the floor of the train is frictionless so that there is no force at your feet, then you will say that the net force on you is $F_{\text{trans}} = -ma$, pointing to the left. You will therefore accelerate with acceleration a to the left, with respect to your frame (the train). In other words, you will remain motionless with respect to the inertial frame of the ground, which is all quite obvious to someone standing on the ground.

In the case where the friction force at your feet is nonzero, but not large enough to balance out the whole $F_{\text{trans}} = -ma$ force, you will end up being jerked toward the back of the train. This undesired motion will continue until you make some adjustments with your feet or hands in order to balance out all of the F_{trans} force. Such adjustments are generally necessary on a subway train, at least here in Boston, where hands are often crucial.

Let's now derive the fictitious forces in their full generality. The main task here is to relate the coordinates in an accelerated frame to those in an inertial frame, so this endeavor will require a bit of math.

9.1 Relating the coordinates

Consider an inertial coordinate system with axes $\hat{\mathbf{x}}_{\text{I}}$, $\hat{\mathbf{y}}_{\text{I}}$, and $\hat{\mathbf{z}}_{\text{I}}$, and let there be another (possibly accelerating) coordinate system with axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. These axes will be allowed to change in an arbitrary manner with respect to the inertial frame. That is, the origin may undergo acceleration, and the axes may rotate. (This is the most general possible motion, as we saw in Section 8.1.) These axes may be considered to be functions of the inertial axes. Let O_{I} and O be the origins of the two coordinate systems.

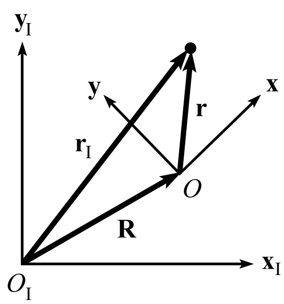

Figure 9.1

Let the vector from O_{I} to O be $\mathbf{R}$. Let the vector from O_{I} to a given particle be $\mathbf{r}_{\text{I}}$. And let the vector from O to the particle be $\mathbf{r}$. (See Fig. 9.1 for the 2-D case.) Then

$$\mathbf{r}_{\text{I}} = \mathbf{R} + \mathbf{r}. \tag{9.1}$$

These vectors have an existence that is independent of any specific coordinate system, but let us write them in terms of some definite coordinates. We may write

$$\begin{aligned}
\mathbf{R} &= X\hat{\mathbf{x}}_{\text{I}} + Y\hat{\mathbf{y}}_{\text{I}} + Z\hat{\mathbf{z}}_{\text{I}}, \\
\mathbf{r}_{\text{I}} &= x_{\text{I}}\hat{\mathbf{x}}_{\text{I}} + y_{\text{I}}\hat{\mathbf{y}}_{\text{I}} + z_{\text{I}}\hat{\mathbf{z}}_{\text{I}}, \\
\mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.
\end{aligned} \tag{9.2}$$

For reasons that will become clear, we have chosen to write $\mathbf{R}$ and $\mathbf{r}_{\text{I}}$ in terms of the inertial-frame coordinates, and $\mathbf{r}$ in terms of the accelerated-frame coordinates. If desired, eq. (9.1) may be written as

$$x_{\text{I}}\hat{\mathbf{x}}_{\text{I}} + y_{\text{I}}\hat{\mathbf{y}}_{\text{I}} + z_{\text{I}}\hat{\mathbf{z}}_{\text{I}} = (X\hat{\mathbf{x}}_{\text{I}} + Y\hat{\mathbf{y}}_{\text{I}} + Z\hat{\mathbf{z}}_{\text{I}}) + (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})\,. \tag{9.3}$$

Our goal is to take the second time derivative of eq. (9.1), and to interpret the result in an $\mathbf{F} = m\mathbf{a}$ form. The second derivative of $\mathbf{r}_{\text{I}}$ is simply the acceleration of the particle with respect to the inertial system, and so Newton's second law tells us that $\mathbf{F} = m\ddot{\mathbf{r}}_{\text{I}}$. The second derivative of $\mathbf{R}$ is the acceleration of the origin of the moving system. The second derivative of $\mathbf{r}$ is the tricky part. Changes in $\mathbf{r}$ can come about in two ways. First, the coordinates (x, y, z) of $\mathbf{r}$ (which are measured with respect to the moving axes) may change. And second, the axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ themselves may change. So even if (x, y, z) remain fixed, the $\mathbf{r}$ vector may still change.[1] Let's be quantitative about this.

Calculation of $d^2\mathbf{r}/dt^2$

We should clarify our goal here. We would like to obtain $d^2\mathbf{r}/dt^2$ in terms of the coordinates in the *moving* frame, because we want to be able to work entirely in terms of the coordinates of the accelerated frame, so that a person in this frame can write down an $\mathbf{F} = m\mathbf{a}$ equation in terms of only her coordinates, without having to consider the underlying inertial frame at all.[2]

The following exercise in taking derivatives works for a general vector $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$ in the moving frame (it's not necessary that it be a position vector). So we'll work with a general $\mathbf{A}$ and then set $\mathbf{A} = \mathbf{r}$ when we're done.

To take d/dt of $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$, we can use the product rule to obtain

$$\frac{d\mathbf{A}}{dt} = \left(\frac{dA_x}{dt}\hat{\mathbf{x}} + \frac{dA_y}{dt}\hat{\mathbf{y}} + \frac{dA_z}{dt}\hat{\mathbf{z}}\right) + \left(A_x\frac{d\hat{\mathbf{x}}}{dt} + A_y\frac{d\hat{\mathbf{y}}}{dt} + A_z\frac{d\hat{\mathbf{z}}}{dt}\right). \tag{9.4}$$

Yes, the product rule works with vectors too. We're doing nothing more than expanding $(A_x + dA_x)(\hat{\mathbf{x}} + d\hat{\mathbf{x}}) - A_x\hat{\mathbf{x}}$, etc., to first order.

The first of the two terms in eq. (9.4) is simply the rate of change of $\mathbf{A}$, as measured with respect to the moving frame. We will denote this quantity by $\delta\mathbf{A}/\delta t$.

The second term arises because the coordinate axes are moving. In what manner are they moving? We have already extracted the motion of the origin of the moving system (by introducing the vector $\mathbf{R}$), so the only thing left is a rotation about some axis $\boldsymbol{\omega}$ through the origin (see Theorem 8.1). This axis may be changing in time, but at any instant a unique axis of rotation describes the system. The fact that the axis may change will be relevant in finding the second derivative of $\mathbf{r}$, but not in finding the first derivative.

We saw in Theorem 8.2 that a vector $\mathbf{B}$ of fixed length (the coordinate axes here do indeed have fixed length), rotating with angular velocity $\boldsymbol{\omega} \equiv \omega\hat{\boldsymbol{\omega}}$, changes at a rate $d\mathbf{B}/dt = \boldsymbol{\omega} \times \mathbf{B}$. In particular, $d\hat{\mathbf{x}}/dt = \boldsymbol{\omega} \times \hat{\mathbf{x}}$, etc. So in eq. (9.4), the $A_x(d\hat{\mathbf{x}}/dt)$ term, for example, equals $A_x(\boldsymbol{\omega} \times \hat{\mathbf{x}}) = \boldsymbol{\omega} \times (A_x\hat{\mathbf{x}})$. Adding on the y and z terms gives $\boldsymbol{\omega} \times (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) = \boldsymbol{\omega} \times \mathbf{A}$. Therefore, eq. (9.4) yields

$$\frac{d\mathbf{A}}{dt} = \frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega} \times \mathbf{A}. \tag{9.5}$$

This agrees with the result obtained in Section 8.5, eq. (8.39). We've basically given the same proof here, but with a little more mathematical rigor.

[1] Remember, the $\mathbf{r}$ vector is *not* simply the ordered triplet (x, y, z). It is the whole expression, $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. So even if (x, y, z) are fixed, meaning that $\mathbf{r}$ doesn't change with respect to the moving system, $\mathbf{r}$ can still change with respect to the inertial system if the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ axes are themselves moving.

[2] In terms of the *inertial* frame, $d^2\mathbf{r}/dt^2$ is simply $d^2(\mathbf{r}_{\mathrm{I}} - \mathbf{R})/dt^2$, but this is not very enlightening by itself.

We still have to take one more time derivative. The time derivative of eq. (9.5) yields

$$\frac{d^2\mathbf{A}}{dt^2} = \frac{d}{dt}\left(\frac{\delta\mathbf{A}}{\delta t}\right) + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{A} + \boldsymbol{\omega} \times \frac{d\mathbf{A}}{dt}\,. \tag{9.6}$$

Applying eq. (9.5) to the first term (with $\delta\mathbf{A}/\delta t$ in place of $\mathbf{A}$), and plugging eq. (9.5) into the third term, gives

$$\begin{aligned}\frac{d^2\mathbf{A}}{dt^2} &= \left(\frac{\delta^2\mathbf{A}}{\delta t^2} + \boldsymbol{\omega} \times \frac{\delta\mathbf{A}}{\delta t}\right) + \left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{A}\right) + \left(\boldsymbol{\omega} \times \frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{A})\right) \\ &= \frac{\delta^2\mathbf{A}}{\delta t^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{A}) + 2\boldsymbol{\omega} \times \frac{\delta\mathbf{A}}{\delta t} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{A}. \end{aligned} \tag{9.7}$$

At this point, we will now set $\mathbf{A} = \mathbf{r}$, so we have

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\delta^2\mathbf{r}}{\delta t^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}, \tag{9.8}$$

where $\mathbf{v} \equiv \delta\mathbf{r}/\delta t$ is the velocity of the particle, as measured with respect to the moving frame.

9.2 The fictitious forces

From eq. (9.1) we have

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}_{\mathrm{I}}}{dt^2} - \frac{d^2\mathbf{R}}{dt^2}\,. \tag{9.9}$$

Let us equate this expression for $d^2\mathbf{r}/dt^2$ with the one in eq. (9.8), and then multiply through by the mass m of the particle. Recognizing that the $m(d^2\mathbf{r}_{\mathrm{I}}/dt^2)$ term is simply the force $\mathbf{F}$ acting on the particle ($\mathbf{F}$ may be gravity, a normal force, friction, tension, etc.), we may write the result as

$$\begin{aligned} m\frac{\delta^2\mathbf{r}}{\delta t^2} &= \mathbf{F} - m\frac{d^2\mathbf{R}}{dt^2} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} \\ &\equiv \mathbf{F} + \mathbf{F}_{\text{translation}} + \mathbf{F}_{\text{centrifugal}} + \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{azimuthal}}, \end{aligned} \tag{9.10}$$

where the *fictitious forces* are defined as

$$\begin{aligned} \mathbf{F}_{\text{trans}} &\equiv -m\frac{d^2\mathbf{R}}{dt^2}\,, \\ \mathbf{F}_{\text{cent}} &\equiv -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \\ \mathbf{F}_{\text{cor}} &\equiv -2m\boldsymbol{\omega} \times \mathbf{v}, \\ \mathbf{F}_{\text{az}} &\equiv -m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}. \end{aligned} \tag{9.11}$$

We have taken the liberty of calling these quantities “forces”, because the left-hand side of eq. (9.10) is simply m times the acceleration, as measured by someone in the accelerated frame. This person should therefore be able to interpret the right-hand side as some effective force. In other words, if a person in the accelerated frame wishes to calculate $m\mathbf{a}_{\text{acc}} \equiv m(\delta^2\mathbf{r}/\delta t^2)$, she simply needs to take the true force $\mathbf{F}$, and then add on all the other terms on the right-hand side, which she will then quite reasonably interpret as forces (in her frame). She will interpret eq. (9.10) as an $\mathbf{F} = m\mathbf{a}$ statement in the form,

$$m\mathbf{a}_{\text{acc}} = \sum \mathbf{F}_{\text{acc}}. \tag{9.12}$$

Note that the extra terms in eq. (9.10) are not actual forces. The constituents of $\mathbf{F}$ are the only real forces in the problem. All we are saying is that if our friend in the moving frame assumes the extra terms are real forces, and if she then adds them to $\mathbf{F}$, then she will get the correct answer for $m(\delta^2\mathbf{r}/\delta t^2)$, the mass times acceleration in her frame.

For example, consider a box (far away from other objects, in outer space) that accelerates at a rate of $g = 10\,\text{m/s}^2$ in some direction. A person in the box will feel a fictitious force of $\mathbf{F}_{\text{trans}} = mg$ down into the floor. For all she knows, she is in a box on the surface of the earth. If she performs various experiments under this assumption, the results will always be what she expects. The surprising fact that no local experiment can distinguish between the fictitious force in the accelerated box and the real gravitational force on the earth is what led Einstein to his Equivalence Principle and his theory of General Relativity (discussed in Chapter 13). These fictitious forces are more meaningful than you might think.

As Einstein explored elevators,
And studied the spinning ice-skaters,
He eyed as suspicious,
The forces, fictitious,
Of gravity's great imitators.

Let's now look at each of the fictitious forces in detail. The translational and centrifugal forces are fairly easy to understand. The Coriolis force is a little more difficult. And the azimuthal force can be easy or difficult, depending on how exactly $\boldsymbol{\omega}$ is changing (we'll mainly deal with the easy case).

9.2.1 Translation force: $-md^2\mathbf{R}/dt^2$

This is the most intuitive of the fictitious forces. We've already discussed this force in the train example in the introduction to this chapter. If $\mathbf{R}$ is the position of the train, then $\mathbf{F}_{\text{trans}} \equiv -m\,d^2\mathbf{R}/dt^2$ is the fictitious force you feel in the accelerated frame.

9.2.2 Centrifugal force: $-m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$

This force goes hand-in-hand with the $mv^2/r = mr\omega^2$ centripetal acceleration as viewed by someone in an inertial frame.

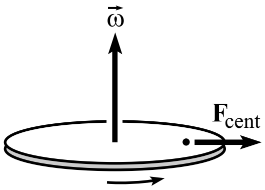

Figure 9.2

Example 1 (Standing on a carousel): Consider a person standing motionless on a carousel. Let the carousel rotate in the x-y plane with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ (see Fig. 9.2). What is the centrifugal force felt by a person standing at a distance r from the center?

Solution: $\boldsymbol{\omega} \times \mathbf{r}$ has magnitude ωr and points in the tangential direction, in the direction of motion ($\boldsymbol{\omega} \times \mathbf{r}$ is simply the velocity as viewed by someone on the ground, after all). Therefore, $m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ has magnitude $mr\omega^2$ and points radially inward. Hence, the centrifugal force, $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, points radially outward with magnitude $mr\omega^2$.

Remark: If the person is not moving with respect to the carousel, and if $\boldsymbol{\omega}$ is constant, then the centrifugal force is the only non-zero fictitious force in eq. (9.10). Since the person is not accelerating in her rotating frame, the net force (as measured in her frame) must be zero. The forces in her frame are (1) gravity pulling downward, (2) a normal force pushing upward (which cancels the gravity), (3) a friction force pushing inward at her feet, and (4) the centrifugal force pulling outward. We conclude that the last two of these must cancel. That is, the friction force points inward with magnitude $mr\omega^2$.

Of course, someone standing on the ground will observe only the first three of these forces, so the net force will not be zero. And indeed, there is a centripetal acceleration, $v^2/r = r\omega^2$, due to the friction force. To sum up: in the inertial frame, the friction force exists to provide an acceleration. In the rotating frame, the friction force exists to balance out the mysterious new centrifugal force, in order to yield zero acceleration. ♣

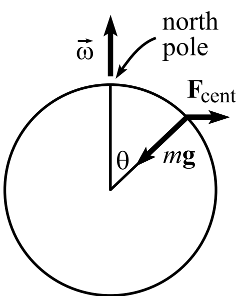

Figure 9.3

Example 2 (Effective gravity force, $m\mathbf{g}_{\text{eff}}$): Consider a person standing motionless on the earth, at a polar angle θ. (See Fig. 9.3. The way we've defined it, θ equals $\pi/2$ minus the latitude angle.) She will feel a force due to gravity, $m\mathbf{g}$, directed toward the center of the earth.[3] But in her rotating frame, she will also feel a centrifugal force, directed away from the rotation axis. The sum of these two forces (that is, what she thinks is gravity) will not point radially, unless she is at the equator or at a pole. Let us denote the sum of these forces as $m\mathbf{g}_{\text{eff}}$.

To calculate $m\mathbf{g}_{\text{eff}}$, we must calculate $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. The $\boldsymbol{\omega} \times \mathbf{r}$ part has magnitude $R\omega \sin\theta$, where R is the radius of the earth, and it is directed tangentially along the latitude circle of radius $R\sin\theta$. So $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points outward from the z-axis, with magnitude $mR\omega^2 \sin\theta$, which is just what we expect for something traveling at frequency ω in a circle of radius $R\sin\theta$. Therefore, the effective gravitational force,

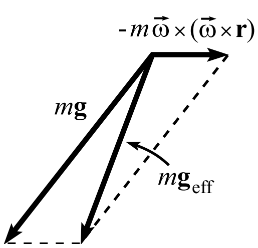

Figure 9.4

$$m\mathbf{g}_{\text{eff}} \equiv m\Big(\mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\Big), \tag{9.13}$$

points slightly in the southerly direction (for someone in the northern hemisphere), as shown in Fig. 9.4. The magnitude of the correction term, $mR\omega^2 \sin\theta$, is small compared to g. Using $\omega \approx 7.3 \cdot 10^{-5}\,\text{s}^{-1}$ (that is, one revolution per day, which is 2π radians per 86,400 seconds) and $R \approx 6.4 \cdot 10^6\,\text{m}$, we find $R\omega^2 \approx .03\,\text{m/s}^2$. Therefore, the correction to g is about 0.3% at the equator. But it is zero at the poles. Exercise 1 and Problem 1 deal further with $\mathbf{g}_{\text{eff}}$.

[3] Note that we are using $\mathbf{g}$ to denote the acceleration due solely to the gravitational force. This isn't the "g-value" that the person measures, as we will shortly see.

REMARKS: In the construction of buildings, and in similar matters, it is of course $\mathbf{g}_{\text{eff}}$, and not $\mathbf{g}$, that determines the "upward" direction in which the building should point. The exact direction to the center of the earth is irrelevant. A plumb bob hanging from the top of a skyscraper touches exactly at the base. Both the bob and the building point in a direction slightly different from the radial, but no one cares.

If you look in table and find that the acceleration due to gravity in New York is $9.803\,\text{m/s}^2$, remember that this is the g_{eff} value and not the g value (which describes only the gravitational force, in our terminology). The way we've defined it, the g value is the acceleration with which things would fall if the earth kept its same shape but somehow stopped spinning. The exact value of g is therefore generally irrelevant. ♣

9.2.3 Coriolis force: $-2m\vec{\omega} \times \mathbf{v}$

While the centrifugal force is very intuitive concept (everyone has gone around a corner in a car), the same thing cannot be said about the Coriolis force. This force requires a non-zero velocity $\mathbf{v}$ relative to the accelerated frame, and people normally don't move appreciably with respect to their car while rounding a corner. To get a feel for this force, let's look at two special cases.

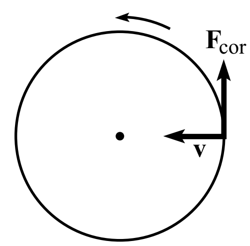

Figure 9.5

Case 1 (Moving radially on a carousel): A carousel rotates with constant angular speed ω. Consider someone walking radially inward on the carousel, with speed v (relative to the carousel) at radius r (see Fig. 9.5). $\boldsymbol{\omega}$ points out of the page, so the Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}$ points tangentially in the direction of the motion of the carousel (that is, to the person's right, in our scenario), with magnitude

$$F_{\text{cor}} = 2m\omega v. \tag{9.14}$$

Let's assume that the person counters this force with a tangential friction force of $2m\omega v$ (pointing to his left) at his feet, so that he continues to walk on the same radial line.[4]

Why does this Coriolis force (and hence the tangential friction force) exist? It exists so that the resultant friction force changes the angular momentum of the person (measured with respect to the lab frame) in the proper way. To see this, take d/dt of $L = mr^2\omega$, where ω is the person's angular speed with respect to the lab frame, which is also the carousel's angular speed. Using $dr/dt = -v$, we have

$$\frac{dL}{dt} = -2mr\omega v + mr^2(d\omega/dt). \tag{9.15}$$

But $d\omega/dt = 0$, because the person is staying on one radial line, and we're assuming that the carousel is arranged to keep a constant ω. Eq. (9.15) then gives $dL/dt = -2mr\omega v$. So the L of the person changes at a rate of $-(2m\omega v)r$. This is simply the radius times the tangential friction force applied by the carousel. In other words, it is the torque applied to the person.

[4]There is also the centrifugal force, which is countered by a radial friction force at the person's feet. This effect won't be important here.

REMARK: What if the person does not apply a tangential friction force at his feet? Then the Coriolis force of $2m\omega v$ produces a tangential acceleration of $2\omega v$ in his frame, and hence the lab frame also. This acceleration exists essentially to keep the angular momentum (measured with respect to the lab frame) of the person constant. (It *is* constant in this scenario, because there are no tangential forces in the lab frame.) To see that this tangential acceleration is consistent with conservation of angular momentum, set $dL/dt = 0$ in eq. (9.15) to obtain $2\omega v = r(d\omega/dt)$. The right-hand side of this is by definition the tangential acceleration. Therefore, saying that L is conserved is the same as saying that $2\omega v$ is the tangential acceleration (for this situation where the inward radial speed is v). ♣

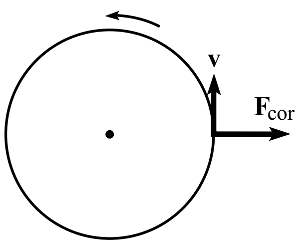

Figure 9.6

Case 2 (Moving tangentially on a carousel): Now consider someone walking tangentially on a carousel, in the direction of the carousel's motion, with speed v (relative to the carousel) at constant radius r (see Fig. 9.6). The Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}$ points radially outward with magnitude $2m\omega v$. Assume that the person applies the friction force necessary to continue moving at radius r.

There is a simple way to see why this outward force of $2m\omega v$ exists. Let $V \equiv \omega r$ be the speed of a point on the carousel at radius r, as viewed by an outside observer. If the person moves tangentially (in the same direction as the spinning) with speed v relative to the carousel, then his speed as viewed by the outside observer is $V + v$. The outside observer therefore sees the person walking in a circle of radius r at speed $V + v$. The acceleration of the person in the ground frame is therefore $(V + v)^2/r$. This acceleration must be caused by an inward-pointing friction force at the person's feet, so

$$F_{\text{friction}} = \frac{m(V+v)^2}{r} = \frac{mV^2}{r} + \frac{2mVv}{r} + \frac{mv^2}{r}. \tag{9.16}$$

This friction force is of course the same in any frame. How, then, does our person on the carousel interpret the three pieces of the inward-pointing friction force in eq. (9.16)? The first term simply balances the outward centrifugal force due to the rotation of the frame, which he always feels. The third term is simply the inward force his feet must apply if he is to walk in a circle of radius r at speed v, which is exactly what he is doing in the rotating frame. The middle term is the additional inward friction force he must apply to balance the outward Coriolis force of $2m\omega v$ (using $V \equiv \omega r$).

Said in an equivalent way, the person on the carousel will write down an $F = ma$ equation of the form (taking radially inward to be positive),

$$\begin{aligned} m\frac{v^2}{r} &= \frac{m(V+v)^2}{r} - \frac{mV^2}{r} - \frac{2mVv}{r}, \qquad \text{or} \\ m\mathbf{a} &= \mathbf{F}_{\text{friction}} + \mathbf{F}_{\text{cent}} + \mathbf{F}_{\text{cor}}. \end{aligned} \tag{9.17}$$

We see that the net force he feels is indeed equal to his ma (where a is measured with respect to his rotating frame).

For cases in between the two special ones above, things aren't so clear, but that's the way it goes. Note that no matter what direction you move on a carousel, the Coriolis force always points in the same perpendicular direction relative to your motion. Whether it's to the right or to the left depends on the direction of the rotation. But given $\boldsymbol{\omega}$, you're stuck with the same relative direction of force.

On a merry-go-round in the night,
Coriolis was shaken with fright.
Despite how he walked,
'Twas like he was stalked
By some fiend always pushing him right.

Let's do some more examples...

Example 1 (Dropped ball): A ball is dropped from a height h, at a polar angle θ (measured down from the north pole). How far to the east is the ball deflected, by the time it hits the ground?

Solution: Note that the ball is indeed deflected to the east, independent of which hemisphere it is in. The angle between $\boldsymbol{\omega}$ and $\mathbf{v}$ is $\pi - \theta$, so the Coriolis force, $-2m\boldsymbol{\omega} \times \mathbf{v}$, is directed eastward with magnitude $2m\omega v \sin\theta$, where $v = gt$ is the speed at time t (t runs from 0 to the usual $\sqrt{2h/g}$).[5] The eastward acceleration at time t is therefore $2\omega gt \sin\theta$. Integrating this to obtain the eastward speed (with an initial eastward speed of 0) gives $v_{\text{east}} = \omega g t^2 \sin\theta$. Integrating once more to obtain the eastward deflection (with an initial eastward deflection of 0) gives $d_{\text{east}} = \omega g t^3 \sin\theta/3$. Plugging in $t = \sqrt{2h/g}$ gives

$$d_{\text{east}} = h\left(\frac{2\sqrt{2}}{3}\right)\left(\omega\sqrt{\frac{h}{g}}\right)\sin\theta. \tag{9.18}$$

This is valid up to second-order effects in the small dimensionless quantity $\omega\sqrt{h/g}$. For an everyday value of h, this quantity is indeed small, since $\omega \approx 7.3 \cdot 10^{-5}\,\text{s}^{-1}$.

Example 2 (Foucault's pendulum): This is the classic example of a consequence of the Coriolis force. It unequivocally shows that the earth rotates. The basic idea is that due to the rotation of the earth, the plane of a swinging pendulum rotates slowly, with a calculable frequency.

In the special case where the pendulum is at one of the poles, this rotation is easy to understand. Consider the north pole. An external observer, hovering above the north pole and watching the earth rotate, sees the pendulum's plane stay fixed (with respect to the distant stars) while the earth rotates counterclockwise beneath it.[6] Therefore, to an observer on the earth, the pendulum's plane rotates clockwise (viewed from above). The frequency of this rotation is of course just the frequency of the earth's rotation, so the earth-based observer sees the pendulum's plane make one revolution each day.

What if the pendulum is not at one of the poles? What is the frequency of the precession? Let the pendulum be located at a polar angle θ. We will work in the approximation where the velocity of the pendulum bob is horizontal. This is essentially

[5] Technically, $v = gt$ isn't quite correct. Due to the Coriolis force, the ball will pick up a small sideways velocity component (this is the point of the problem). This component will then produce a second-order Coriolis force that effects the vertical speed (see Exercise 2). We may, however, ignore this small effect in this problem.

[6] Assume that the pivot of the pendulum is a frictionless bearing, so that it can't provide any torque to twist the pendulum's plane.

true if the pendulum's string is very long; the correction due to the rising and falling of the bob is negligible. The Coriolis force, $-2m\boldsymbol{\omega} \times \mathbf{v}$, points in some complicated direction, but fortunately we are concerned only with the component that lies in the horizontal plane. The vertical component serves only to modify the apparent force of gravity and is therefore negligible. (Although the frequency of the pendulum depends on g, the resulting modification is very small.)

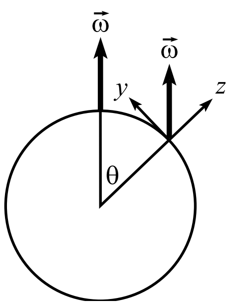

Figure 9.7

With this in mind, let's break $\boldsymbol{\omega}$ into vertical and horizontal components in a coordinate system located at the pendulum. From Fig. 9.7, we see that

$$\boldsymbol{\omega} = \omega\cos\theta\hat{\mathbf{z}} + \omega\sin\theta\hat{\mathbf{y}}. \tag{9.19}$$

We'll ignore the y-component, because it produces a Coriolis force in the $\hat{\mathbf{z}}$ direction (since $\mathbf{v}$ lies in the horizontal x-y plane). So for our purposes, $\boldsymbol{\omega}$ is essentially equal to $\omega\cos\theta\hat{\mathbf{z}}$. From this point on, the problem of finding the frequency of precession can be done in numerous ways. We'll present two solutions.

First solution (The slick way): The horizontal component of the Coriolis force has magnitude

$$F_{\text{cor}}^{\text{horiz}} = |-2m(\omega\cos\theta\hat{\mathbf{z}}) \times \mathbf{v}| = 2m(\omega\cos\theta)v, \tag{9.20}$$

and it is perpendicular to $\mathbf{v}(t)$. Therefore, as far as the pendulum is concerned, it is located at the north pole of a planet called Terra Costhetica which has rotational frequency $\omega\cos\theta$. But as we saw above, the precessional frequency of a Foucault pendulum located at the north pole of such a planet is simply

$$\omega_F = \omega\cos\theta, \tag{9.21}$$

in the clockwise direction. So that's our answer.[7]

Second solution (In the pendulum's frame): Let's work in the frame of the vertical plane that the Foucault pendulum sweeps through. Our goal is to find the rate of precession of this frame. With respect to a frame fixed on the earth (with axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$), we know that this plane rotates with frequency $\boldsymbol{\omega}_F = -\omega\hat{\mathbf{z}}$ if we're at the north pole ($\theta = 0$), and frequency $\boldsymbol{\omega}_F = 0$ if we're at the equator ($\theta = \pi/2$). So if there's any justice in the world, the general answer has got to be $\boldsymbol{\omega}_F = -\omega\cos\theta\hat{\mathbf{z}}$, and that's what we'll now show.

Working in the frame of the plane of the pendulum is useful, because we can take advantage of the fact that *the pendulum feels no sideways forces in this frame*, because otherwise it would move outside of the plane (which it doesn't, by definition).

The frame fixed on the earth rotates with frequency $\boldsymbol{\omega} = \omega\cos\theta\hat{\mathbf{z}} + \omega\sin\theta\hat{\mathbf{y}}$, with respect to the inertial frame. Let the pendulum's plane rotate with frequency $\boldsymbol{\omega}_F = \omega_F\hat{\mathbf{z}}$ with respect to the earth frame. Then the angular velocity of the pendulum's frame with respect to the inertial frame is

$$\boldsymbol{\omega} + \boldsymbol{\omega}_F = (\omega\cos\theta + \omega_F)\hat{\mathbf{z}} + \omega\sin\theta\hat{\mathbf{y}}. \tag{9.22}$$

To find the horizontal component of the Coriolis force in this rotating frame, we only care about the $\hat{\mathbf{z}}$ part of this frequency. The horizontal Coriolis force therefore has magnitude $2m(\omega\cos\theta + \omega_F)v$. But in the frame of the pendulum, there is no horizontal force, so this must be zero. Therefore,

$$\omega_F = -\omega\cos\theta. \tag{9.23}$$

[7]As mentioned above, the setup isn't *exactly* like the one on the new planet. There will be a vertical component of the Coriolis force for the pendulum on the earth, but this effect is negligible.

This agrees with eq. (9.21), where we just wrote down the magnitude of ω_F.

9.2.4 Azimuthal force: $-m(d\boldsymbol{\omega}/dt) \times \mathbf{r}$

In this section, we will restrict ourselves to the simple and intuitive case where $\boldsymbol{\omega}$ changes only in magnitude (that is, not in direction).[8] In this case, the azimuthal force may be written as

$$\mathbf{F}_{\text{az}} = -m\dot{\omega}\hat{\boldsymbol{\omega}} \times \mathbf{r}. \tag{9.24}$$

This force is easily understood by considering a person standing at rest with respect to a rotating carousel. If the carousel speeds up, then the person must feel a tangential friction force at his feet if he is to remain fixed on the carousel. This friction force equals ma_{tan}, where $a_{\text{tan}} = r\dot{\omega}$ is the tangential acceleration as measured in the ground frame. But from the person's point of view in the rotating frame, he is not moving, so there must be some other mysterious force that balances the friction. This is the azimuthal force. Quantitatively, when $\hat{\boldsymbol{\omega}}$ is orthogonal to $\mathbf{r}$, we have $|\hat{\boldsymbol{\omega}} \times \mathbf{r}| = r$, so the azimuthal force in eq. (9.24) has magnitude $mr\dot{\omega}$. This is the same as the magnitude of the friction force, as it should be.

What we have here is exactly the same effect that we had with the translation force on the accelerating train. If the floor speeds up beneath you, then you must apply a friction force if you don't want to be thrown backwards with respect to the floor. If you shut your eyes and ignore the centrifugal force, then you can't tell if you're on a linearly accelerating train, or on an angularly accelerating carousel. The translation and azimuthal forces both arise from the acceleration of the floor. (Well, for that matter, the centrifugal force does, too.)

We can also view things in terms of rotational quantities, as opposed to the linear a_{tan} acceleration above. If the carousel speeds up, then a torque must be applied to the person if he is to remain fixed on the carousel, because his angular momentum in the fixed frame increases. Therefore, he must feel a friction force at his feet.

Let's show that this friction force, which produces the change in angular momentum of the person in the fixed frame, exactly cancels the azimuthal force in the rotating frame, thereby yielding zero net force in the rotating frame.[9] Since $L = mr^2\omega$, we have $dL/dt = mr^2\dot{\omega}$ (assuming r is fixed). And since $dL/dt = \tau = rF$, we see that the required friction force is $F = mr\dot{\omega}$. And as we saw above, when $\hat{\boldsymbol{\omega}}$ is orthogonal to $\mathbf{r}$, the azimuthal force in eq. (9.24) also equals $mr\dot{\omega}$, in the direction opposite to the carousel's motion.

Example (Spinning ice skater): We have all seen ice skaters increase their angular speed by bringing their arms in close to their body. This is easily understood in terms of angular momentum; a smaller moment of inertia requires a larger ω, to keep

[8] The more complicated case where $\boldsymbol{\omega}$ changes direction is left for Problem 9.

[9] This is basically the same calculation as the one above, with an extra r thrown in.

L constant. But let's analyze the situation here in terms of fictitious forces. We will idealize things by giving the skater massive hands at the end of massless arms attached to a massless body.[10] Let the hands have total mass m, and let them be drawn in radially.

Look at things in the skater's frame (which has an increasing ω), defined by the vertical plane containing the hands. The crucial thing to realize is that the skater always remains in the skater's frame (a fine tautology, indeed). Therefore, the skater must feel zero net tangential force in her frame, because otherwise she would accelerate with respect to it. Her hands are being drawn in by a muscular force that works against the centrifugal force, but there is no net tangential force on the hands in the skater's frame.

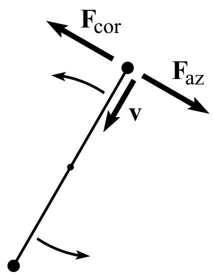

Figure 9.8

What are the tangential forces in the skater's frame? (See Fig. 9.8.) Let the hands be drawn in at speed v. Then there is a Coriolis force (in the same direction as the spinning) with magnitude $2m\omega v$. There is also an azimuthal force with magnitude $mr\dot{\omega}$ (in the direction opposite the spinning, as you can check). Since the net tangential force is zero in the skater's frame, we must have

$$2m\omega v = mr\dot{\omega}. \tag{9.25}$$

Does this relation make sense? Well, let's look at things in the ground frame. The total angular momentum of the hands in the ground frame is constant. Therefore, $d(mr^2\omega)/dt = 0$. Taking this derivative and using $dr/dt \equiv -v$ gives eq. (9.25).

A word of advice about using fictitious forces: Decide which frame you are going to work in (the lab frame or the accelerated frame), and then stick with it. The common mistake is to work a little in one frame and a little on the other, without realizing it. For example, you might introduce a centrifugal force on someone sitting at rest on a carousel, but then also give her a centripetal acceleration. This is incorrect. In the lab frame, there is a centripetal acceleration (caused by the friction force) and no centrifugal force. In the rotating frame, there is a centrifugal force (which cancels the friction force) and no centripetal acceleration (because the person is sitting at rest on the carousel, consistent with the fact that the net force is zero). Basically, if you ever mention the words "centrifugal" or "Coriolis", etc., then you had better be working in an accelerated frame.

[10] This reminds me of a joke about a spherical cow...

9.3 Exercises

Section 9.2: The fictitious forces

1. **Magnitude of $\mathbf{g}_{\text{eff}}$** $*$

   What is the magnitude of $\mathbf{g}_{\text{eff}}$? Give you answer to the leading-order correction in ω.

2. **Corrections to gravity** $**$

   A mass is dropped from rest from a point directly above the equator. Let the initial distance from the center of the earth be R, and let the distance fallen be d. If we consider only the centrifugal force, then the correction to g is $\omega^2(R-d)$. There is, however, also a second-order Coriolis effect. What is the sum of these corrections?[11]

3. **Southern deflection** $**$

   A ball is dropped from a height h (small compared to the radius of the earth), at a polar angle θ. How far to the *south* (in the northern hemisphere) is it deflected away from the $\mathbf{g}_{\text{eff}}$ direction, by the time it hits the ground? (This is a second order Coriolis effect.)

4. **Oscillations across equator** $*$

   A bead lies on a frictionless wire which lies in the north-south direction across the equator. The wire takes the form of an arc of a circle; all points are the same distance from the center of the earth. The bead is released from rest at a short distance from the equator. Because $\mathbf{g}_{\text{eff}}$ does not point directly toward the earth's center, the bead will head toward the equator and undergo oscillatory motion. What is the frequency of these oscillations?

5. **Roche limit** $*$

   Exercise 4.29 dealt with the Roche limit for a particle falling in radially toward a planet. Show that the Roche limit for a object in a circular orbit is

   $$d = R\left(\frac{3\rho_{\text{p}}}{\rho_{\text{r}}}\right)^{1/3}. \tag{9.26}$$

6. **Spinning bucket** $**$

   An upright bucket of water is spun at frequency ω around the vertical axis. If the water is at rest with respect to the bucket, find the shape of the water's surface.

7. **Coin on turntable** $***$

   A coin stands upright at an arbitrary point on a rotating turntable, and rotates (without slipping) with the required speed to make its center remain motionless

[11] g will also vary with height, but let's not worry about that here.

in the lab frame. In the frame of the turntable, the coin will roll around in a circle with the same frequency as that of the turntable. In the frame of the turntable, show that

(a) $\mathbf{F} = d\mathbf{p}/dt$, and

(b) $\boldsymbol{\tau} = d\mathbf{L}/dt$.

8. **Precession viewed from rotating frame** ***

Consider a top made of a wheel with all its mass on the rim. A massless rod (perpendicular to the plane of the wheel) connects the CM to the pivot. Initial conditions have been set up so that the top undergoes precession, with the rod always horizontal.

In the language of Figure 8.27, we may write the angular velocity of the top as $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3$ (where $\hat{\mathbf{x}}_3$ is horizontal here). Consider things in the frame rotating around the $\hat{\mathbf{z}}$-axis with angular speed Ω. In this frame, the top spins with angular speed ω' around its *fixed* symmetry axis. Therefore, in this frame $\boldsymbol{\tau} = 0$, because there is no change in $\mathbf{L}$.

Verify explicitly that $\boldsymbol{\tau} = 0$ (calculated with respect to the pivot) in this rotating frame (you will need to find the relation between ω' and Ω). In other words, show that the torque due to gravity is exactly canceled by the torque do to the Coriolis force. (You can easily show that the centrifugal force provides no net torque.)

9.4 Problems

Section 9.2: The fictitious forces

1. **$\mathbf{g}_{\text{eff}}$ vs. $\mathbf{g}$** $*$

   For what θ is the angle between $m\mathbf{g}_{\text{eff}}$ and $\mathbf{g}$ maximum?

2. **Longjumping in $\mathbf{g}_{\text{eff}}$** $*$

   If a longjumper can jump 8 meters at the north pole, how far can he jump at the equator? (Ignore effects of wind resistance, temperature, and runways made of ice. And assume that the jump is made in the north-south direction at the equator, so that there is no Coriolis force.)

3. **Lots of circles** $**$

   (a) Two circles in a plane, C_1 and C_2, each rotate with frequency ω (relative to an inertia frame). See Fig. 9.9. The center of C_1 is fixed in an inertial frame, and the center of C_2 is fixed on C_1. A mass is fixed on C_2. The position of the mass relative to the center of C_1 is $\mathbf{R}(t)$. Find the fictitious force felt by the mass.

   (b) N circles in a plane, C_i, each rotate with frequency ω (relative to an inertia frame). See Fig. 9.10. The center of C_1 is fixed in an inertial frame, and the center of C_i is fixed on C_{i-1} (for $i = 2, \ldots, N$). A mass is fixed on C_N. The position of the mass relative to the center of C_1 is $\mathbf{R}(t)$. Find the fictitious force felt by the mass.

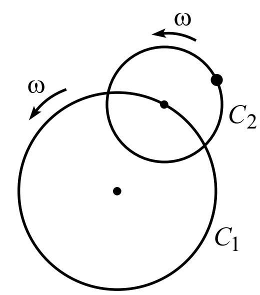

Figure 9.9

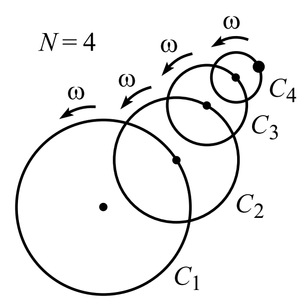

Figure 9.10

4. **Which way down?** $*$

   You are floating high up in a balloon, at rest with respect to the earth. Give three quasi-reasonable definitions for which point on the ground is right "below" you.

5. **Mass on a turntable** $*$

   A mass rests motionless with respect to the lab frame, while a frictionless turntable rotates beneath it. The frequency of the turntable is ω, and the mass is located at radius r. In the frame of the turntable, what are the forces acting on the mass?

6. **Released mass** $*$

   A mass is bolted down to a frictionless turntable. The frequency of rotation is ω, and the mass is located at a radius a. The mass is released. Viewed from an inertial frame, it travels in a straight line. In the rotating frame, what path does the mass take? Specify $r(t)$ and $\theta(t)$, where θ is the angle with respect to the initial radius.

7. **Coriolis circles** $*$

A puck slides with speed v on frictionless ice. The surface is "level", in the sense that it is orthogonal to $\mathbf{g}_{\text{eff}}$ at all points. Show that the puck moves in a circle, as seen in the earth's rotating frame. What is the radius of the circle? What is the frequency of the motion? Assume that the radius of the circle is small compared to the radius of the earth.

8. **Shape of the earth** $***$

The earth bulges slightly at the equator, due to the centrifugal force in the earth's rotating frame. Show that the height of a point on the earth (relative to a spherical earth), is given by

$$h = R\left(\frac{R\omega^2}{6g}\right)(3\sin^2\theta - 2), \tag{9.27}$$

where θ is the polar angle (the angle down from the north pole), and R is the radius of the earth.

9. **Changing $\boldsymbol{\omega}$'s direction** $***$

Consider the special case where a reference frame's $\boldsymbol{\omega}$ changes only in direction (and not in magnitude). In particular, consider a cone rolling on a table, which is a natural example of such a situation.

The instantaneous $\boldsymbol{\omega}$ for a rolling cone is its line of contact with the table, because these are the points that are instantaneously at rest. This line precesses around the origin. Let the frequency of the precession be $\boldsymbol{\Omega}$. Let the origin of the cone frame be the tip of the cone. This point remains fixed in the inertial frame.

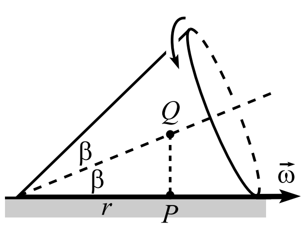

Figure 9.11

In order to isolate the azimuthal force, consider the special case of a point P that lies on the instantaneous $\boldsymbol{\omega}$ and which is motionless with respect to the cone (see Fig. 9.11). From eq. (9.11), we then see that the centrifugal, Coriolis, and translation forces are zero. The only remaining fictitious force is the azimuthal force, and it arises from the fact that P is accelerating up away from the table.

(a) Find the acceleration of P.

(b) Calculate the azimuthal force on a mass m located at P, and show that the result is consistent with part (a).

9.5 Solutions

1. **$\mathbf{g}_{\text{eff}}$ vs. g**

   The forces $m\mathbf{g}$ and $\mathbf{F}_{\text{cent}}$ are shown in Fig. 9.12. The magnitude of $\mathbf{F}_{\text{cent}}$ is $mR\omega^2 \sin\theta$, so the component of $\mathbf{F}_{\text{cent}}$ perpendicular to $m\mathbf{g}$ is $mR\omega^2 \sin\theta\cos\theta = mR\omega^2(\sin 2\theta)/2$. For small $\mathbf{F}_{\text{cent}}$, maximizing the angle between $\mathbf{g}_{\text{eff}}$ and $\mathbf{g}$ is equivalent to maximizing this perpendicular component. Therefore, we obtain the maximum angle when

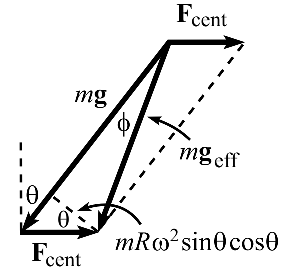

Figure 9.12

$$\theta = \frac{\pi}{4}\,. \tag{9.28}$$

   The maximum angle turns out to be $\phi \approx \sin\phi \approx \Big(mR\omega^2(\sin\frac{\pi}{2})/2\Big)/mg = R\omega^2/2g \approx 0.0017$, which is about 0.1°. For this $\theta = \pi/4$ case, the line along $\mathbf{g}_{\text{eff}}$ misses the center of the earth by about 10 km, as you can show.

   REMARK: The above method works only when the magnitude of $\mathbf{F}_{\text{cent}}$ is much smaller than mg; we dropped higher order terms in the above calculation. One way of solving the problem exactly is to break $\mathbf{F}_{\text{cent}}$ into components parallel and perpendicular to $\mathbf{g}$. If ϕ is the angle between $\mathbf{g}_{\text{eff}}$ and $\mathbf{g}$, then from Fig. 9.12 we have

$$\tan\phi = \frac{mR\omega^2 \sin\theta\cos\theta}{mg - mR\omega^2\sin^2\theta}\,. \tag{9.29}$$

   We can then maximize ϕ by taking a derivative. But be careful if $R\omega^2 > g$, in which case maximizing ϕ doesn't mean maximizing $\tan\phi$. We'll let you work this out, and instead we'll give the following slick geometric solution.

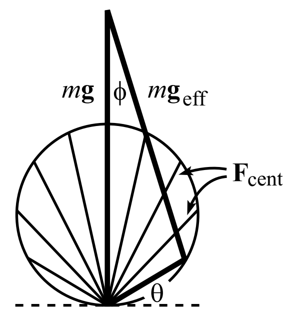

Figure 9.13

   Draw the $\mathbf{F}_{\text{cent}}$ vectors, for various θ, relative to $m\mathbf{g}$. The result looks like Fig. 9.13. Since the lengths of the $\mathbf{F}_{\text{cent}}$ vectors are proportional to $\sin\theta$, you can show that the tips of the $\mathbf{F}_{\text{cent}}$ vectors form a circle. The maximum ϕ is therefore achieved when $\mathbf{g}_{\text{eff}}$ is tangent to this circle, as shown in Fig. 9.14. In the limit where $g \gg R\omega^2$ (that is, in the limit of a small circle), we want the point of tangency to be the rightmost point on the circle, so the maximum ϕ is achieved when $\theta = \pi/4$, in which case $\sin\phi \approx (R\omega^2/2)/g$. But in the general case, Fig. 9.14 shows that the maximum ϕ is given by

$$\sin\phi_{\text{max}} = \frac{\frac{1}{2}mR\omega^2}{mg - \frac{1}{2}mR\omega^2}\,. \tag{9.30}$$

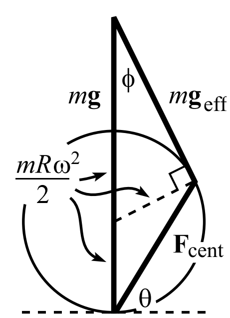

Figure 9.14

   In the limit of small ω, this is approximately $R\omega^2/2g$, as above.

   The above reasoning holds only if $R\omega^2 < g$. In the case where $R\omega^2 > g$ (that is, the circle extends above the top end of the $m\mathbf{g}$ segment), the maximum ϕ is simply π, and it is achieved at $\theta = \pi/2$. ♣

2. **Longjumping in $\mathbf{g}_{\text{eff}}$**

   Let the jumper take off with speed v, at an angle θ. Then $g_{\text{eff}}(t/2) = v\sin\theta$ tells us that the time in the air is $t = 2v\sin\theta/g_{\text{eff}}$. The distance traveled is therefore

$$d = v_x t = vt\cos\theta = \frac{2v^2\sin\theta\cos\theta}{g_{\text{eff}}} = \frac{v^2\sin 2\theta}{g_{\text{eff}}}\,. \tag{9.31}$$

   This is maximum when $\theta = \pi/4$, as we well know. So we see that $d \propto 1/\sqrt{g_{\text{eff}}}$. Taking $g_{\text{eff}} \approx 10\ \text{m/s}^2$ at the north pole, and $g_{\text{eff}} \approx (10 - 0.03)\ \text{m/s}^2$ at the equator, we find that the jump at the equator is approximately 1.0015 times as long as the one on the north pole. So the longjumper gains about one centimeter.

   REMARK: For a longjumper, the optimal angle of takeoff is undoubtedly not $\pi/4$. The act of changing the direction abruptly from horizontal to such a large angle would entail a significant loss in speed. The optimal angle is some hard-to-determine angle less than $\pi/4$. But this won't change our general $d \propto 1/\sqrt{g_{\text{eff}}}$ result (which follows from dimensional analysis), so our answer still holds. ♣

3. **Lots of circles**

   (a) The fictitious force, $\mathbf{F}_\text{f}$, on the mass has an $\mathbf{F}_\text{cent}$ part and an $\mathbf{F}_\text{trans}$ part, because the center of C_2 is moving. So the fictitious force is

   $$\mathbf{F}_\text{f} = m\omega^2 \mathbf{r}_2 + \mathbf{F}_\text{trans}, \tag{9.32}$$

   where $\mathbf{r}_2$ is the position of the mass in the frame of C_2. But $\mathbf{F}_\text{trans}$, which arises from the acceleration of the center of C_2, is simply the centrifugal force felt by any point on C_1. Therefore,

   $$\mathbf{F}_\text{trans} = m\omega^2 \mathbf{r}_1, \tag{9.33}$$

   where $\mathbf{r}_1$ is the position of the center of C_2, in the frame of C_1. Substituting this into eq. (9.32) gives

   $$\begin{aligned} \mathbf{F}_\text{f} &= m\omega^2(\mathbf{r}_2 + \mathbf{r}_1) \\ &= m\omega^2 \mathbf{R}(t). \end{aligned} \tag{9.34}$$

   (b) The fictitious force, $\mathbf{F}_\text{f}$, on the mass has an $\mathbf{F}_\text{cent}$ part and an $\mathbf{F}_\text{trans}$ part, because the center of the Nth circle is moving. So the fictitious force is

   $$\mathbf{F}_\text{f} = m\omega^2 \mathbf{r}_N + \mathbf{F}_{\text{trans},N}, \tag{9.35}$$

   where $\mathbf{r}_N$ is the position of the mass in the frame of C_N. But $\mathbf{F}_{\text{trans},N}$ is simply the centrifugal force felt by a point on the $(N-1)$st circle, plus the translation force coming from the movement of the center of the $(N-1)$st circle. Therefore,

   $$\mathbf{F}_{\text{trans},N} = m\omega^2 \mathbf{r}_{N-1} + \mathbf{F}_{\text{trans},N-1}. \tag{9.36}$$

   Substituting this into eq. (9.35) and successively rewriting the $\mathbf{F}_{\text{trans},i}$ terms in a similar manner, gives

   $$\begin{aligned} \mathbf{F}_\text{f} &= m\omega^2(\mathbf{r}_N + \mathbf{r}_{N-1} + \cdots + \mathbf{r}_1) \\ &= m\omega^2 \mathbf{R}(t). \end{aligned} \tag{9.37}$$

   The main point in this problem is that $\mathbf{F}_\text{cent}$ is linear in $\mathbf{r}$.

REMARK: There is actually a much easier way to see that $\mathbf{F}_\text{f} = m\omega^2 \mathbf{R}(t)$. Since all the circles rotate with the same ω, they may as well be glued together. Such a rigid setup would indeed yield the same ω for all the circles. This is similar to the moon rotating once on its axis for every revolution is makes around the earth, thereby causing the same side to always face the earth. It is then clear that the mass simply moves in a circle at frequency ω, yielding a fictitious centrifugal force of $m\omega^2\mathbf{R}(t)$. And we see that the radius R is in fact constant. ♣

4. **Which way down?**

   Here are three possible definitions of the point "below" you on the ground: (1) The point that lies along the line between you and the center of the earth, (2) The point where a hanging plumb bob rests, and (3) The point where a dropped object hits the ground.

The second definition is the most reasonable, because it defines the upward direction in which buildings are constructed. It differs from the first definition due to the centrifugal force which makes $\mathbf{g}_{\text{eff}}$ point in a slightly different direction from $\mathbf{g}$. The third definition differs from the second due to the Coriolis force. The velocity of the falling object produces a Coriolis force which causes an eastward deflection.

Note that all three definitions are equivalent at the poles. Additionally, definitions 1 and 2 are equivalent at the equator.

5. **Mass on turntable**

In the lab frame, the net force on the mass is zero, because it is sitting at rest. (The normal force cancels the gravitational force.) But in the rotating frame, the mass travels in a circle of radius r, with frequency ω. So in the rotating frame there must be a force of $m\omega^2 r$ inward to account for the centripetal acceleration. And indeed, the mass feels a centrifugal force of $m\omega^2 r$ outward, and a Coriolis force of $2m\omega v = 2m\omega^2 r$ inward, which sum to the desired force (see Fig. 9.15).

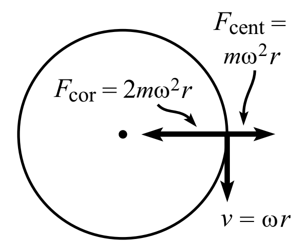

Figure 9.15

Remark: The net inward force in this problem is a little different from that for someone swinging around in a circle in an inertial frame. If a skater maintains a circular path by holding onto a rope whose other end is fixed, she has to use her muscles to maintain the position of her torso with respect to her arm, and her head with respect to her torso, etc. But if a person takes the place of the mass in this problem, she needs to exert no effort to keep her body moving in the circle (which is clear, when looked at from the inertial frame), because each atom in her body is moving at (essentially) the same speed and radius, and therefore feels the same Coriolis and centrifugal forces. So she doesn't really *feel* this force, in the same sense that someone doesn't feel gravity when in free-fall with no air resistance. ♣

6. **Released mass**

Let the x'- and y'-axes of the rotating frame coincide with the x- and y-axes of the inertial frame at the moment the mass is released (at $t = 0$). Let the mass initially be located on the y'-axis. Then after a time t, the situation looks like that in Fig. 9.16. The speed of the mass is $v = a\omega$, so it has traveled a distance $a\omega t$. The angle that its position vector makes with the inertial x-axis is therefore $\tan^{-1}\omega t$, with counterclockwise taken to be positive. Hence, the angle that its position vector makes with the rotating y'-axis is $\theta(t) = -(wt - \tan^{-1}\omega t)$. And the radius is simply $r(t) = a\sqrt{1+\omega^2 t^2}$. So for large t, $r(t) \approx a\omega t$ and $\theta(t) \approx -wt + \pi/2$, which make sense.

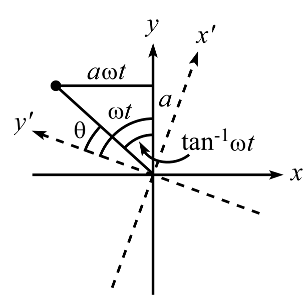

Figure 9.16

7. **Coriolis circles**

By construction, the normal force from the ice exactly cancels all effects of the gravitational and centrifugal forces in the rotating frame of the earth. We therefore need only concern ourselves with the Coriolis force. This force equals $\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$.

Let the angle down from the north pole be θ; we assume the circle is small enough so that θ is essentially constant throughout the motion. Then the component of the Coriolis force that points horizontally along the surface has magnitude $f = 2m\omega v\cos\theta$ and is perpendicular to the direction of motion. (The vertical component of the Coriolis force will simply modify the required normal force.) Because this force is perpendicular to the direction of motion, v does not change. Therefore, f is constant.

But a constant force perpendicular to the motion of a particle produces a circular path. The radius of the circle is given by

$$2m\omega v\cos\theta = \frac{mv^2}{r} \qquad \Longrightarrow \qquad r = \frac{v}{2\omega\cos\theta}\,. \tag{9.38}$$

The frequency of the circular motion is

$$\omega' = \frac{v}{r} = 2\omega\cos\theta. \tag{9.39}$$

REMARKS: To get a rough idea of the size of the circle, you can show (using $\omega \approx 7.3\cdot 10^{-5}\,\mathrm{s}^{-1}$) that $r \approx 10\,\mathrm{km}$ when $v = 1\,\mathrm{m/s}$ and $\theta = 45^\circ$. Even the tiniest bit of friction will clearly make this effect essentially impossible to see.

For the $\theta \approx \pi/2$ (that is, near the equator), the component of the Coriolis force along the surface is negligible, so r becomes large, and ω' goes to 0.

For the $\theta \approx 0$ (that is, near the north pole), the Coriolis force essentially points along the surface. The above equations give $r \approx v/(2\omega)$, and $\omega' \approx 2\omega$. For the special case where the center of the circle is the north pole, this $\omega' \approx 2\omega$ result might seem incorrect, because you might want to say that the circular motion should be achieved by having the puck remain motionless in the inertial fame, while the earth rotates beneath it (thus making $\omega' = \omega$). The error in this reasoning is that the "level" earth is not spherical, due to the non-radial direction of $\mathbf{g}_{\mathrm{eff}}$. If the puck starts out motionless in the inertial frame, it will be drawn toward the north pole, due to the component of the gravitational force along the "level" surface. In order to not fall toward the pole, the puck needs to travel with frequency ω (relative to the inertial frame) in the direction opposite[12] to the earth's rotation.[13] The puck therefore moves at frequency 2ω relative to the frame of the earth. ♣

8. **Shape of the earth**

In the reference frame of the earth, the forces on an atom at the surface are: earth's gravity, the centrifugal force, and the normal force from the ground below it. These three forces must sum to zero. Therefore, the sum of the gravity plus centrifugal forces must be normal to the surface. Said differently, the gravity-plus-centrifugal force must have no component along the surface. Said in yet another way, the potential energy function derived from the gravity-plus-centrifugal force must be constant along the surface. (Otherwise, a piece of the earth would want to move along the surface, which would mean we didn't have the correct surface to begin with.)

If x is the distance from the earth's axis, then the centrifugal force is $F_c = m\omega^2 x$, directed outward. The potential energy function for this force is $V_c = -m\omega^2x^2/2$, up to an arbitrary additive constant. The potential energy for the earth's gravitation force is simply mgh. (We've arbitrarily chosen the original spherical surface have zero potential; any other choice would add on an irrelevant constant. Also, we've assumed that the slight distortion of the earth won't make the mgh result invalid. This is true to lowest order in h/R, which you can demonstrate if you wish.)

The equal-potential condition is therefore

$$mgh - \frac{m\omega^2x^2}{2} = C, \tag{9.40}$$

[12] Of course, the puck could also move with frequency ω in the *same* direction as the earth's rotation. But in this case, the puck simply sits at one place on the earth.

[13] The reason for this is the following. In the rotating frame of the puck, the puck feels the same centrifugal force that it would feel if it were at rest on the earth, spinning with it. The puck therefore happily stays at the same θ value on the "level" surface, just as a puck at rest on the earth does.

where C is a constant to be determined. Using $x = r\sin\theta$, we obtain

$$h = \frac{\omega^2 r^2 \sin^2\theta}{2g} + B, \tag{9.41}$$

where $B \equiv C/(mg)$ is another constant. We may replace the r here with the radius of the earth, R, with negligible error.

Depending what the constant B is, this equation describes a whole family of surfaces. We may determine the correct value of B by demanding that the volume of the earth be the same as it would be in its spherical shape if the centrifugal force were turned off. This is equivalent to demanding that the integral of h over the surface of the earth is zero. The integral of $(a\sin^2\theta + b)$ over the surface of the earth is (the integral is easy if we write $\sin^2\theta$ as $1 - \cos^2\theta$)

$$\begin{aligned}
\int_0^\pi \big(a(1-\cos^2\theta)+b\big)\,2\pi R^2\sin\theta\,d\theta &= \int_0^\pi \big(-a\cos^2\theta + (a+b)\big)\,2\pi R^2 \sin\theta\,d\theta \\
&= 2\pi R^2\left(\frac{a\cos^3\theta}{3} - (a+b)\cos\theta\right)\Big|_0^\pi \\
&= 2\pi R^2\left(-\frac{2a}{3} + 2(a+b)\right). \qquad (9.42)
\end{aligned}$$

Hence, we need $b = -(2/3)a$ for this integral to be zero. Plugging this result into eq. (9.41) gives

$$h = R\left(\frac{R\omega^2}{6g}\right)(3\sin^2\theta - 2), \tag{9.43}$$

as desired.

9. **Changing $\boldsymbol{\omega}$'s direction**

(a) Let Q be the point which lies on the axis of the cone and which is directly above P (see Fig. 9.17). If P is a distance r from the origin, and if the half-angle of the cone is β, then Q is a height $y = r\tan\beta$ above P.

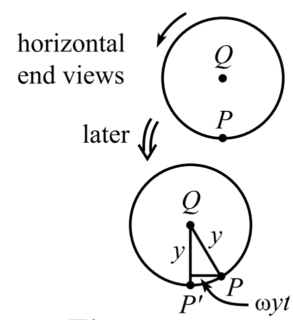

Figure 9.17

Consider the situation an infinitesimal time t later. Let P' be the point that is now directly below Q (see Fig. 9.17). The angular speed of the cone is ω, so Q moves horizontally at a speed $v_{\mathrm{Q}} = \omega y = \omega r\tan\beta$. Therefore, in the infinitesimal time t, we see that Q moves a distance ωyt to the side.

This distance ωyt is also (essentially) the horizontal distance between P and P'. Therefore, a little geometry tells us that P is now a distance

$$h(t) = y - \sqrt{y^2 - (\omega yt)^2} \approx \frac{(\omega t)^2 y}{2} = \frac{1}{2}(\omega^2 y)t^2 \tag{9.44}$$

above the table. Since P started on the table with zero speed, this means that P is undergoing an acceleration of $\omega^2 y$ in the vertical direction. A mass m located at P must therefore feel a force $F_P = m\omega^2 y$ in the upward direction, if it is to remain motionless with respect to the cone.

(b) The precession frequency Ω (which is how fast $\boldsymbol{\omega}$ swings around the origin) is equal to the speed of Q divided by r. This is true because Q is always directly above $\boldsymbol{\omega}$, so it moves in a circle of radius r around the z-axis. Therefore, $\boldsymbol{\Omega}$ has magnitude $v_{\mathrm{Q}}/r = \omega y/r$, and it points in the downward vertical direction (for the situation shown in Fig. 9.11). Hence, $d\boldsymbol{\omega}/dt = \boldsymbol{\Omega}\times\boldsymbol{\omega}$ has magnitude $\omega^2 y/r$, and it points in the horizontal direction (out of the page). Therefore,

$\mathbf{F}_{\text{az}} = -m(d\boldsymbol{\omega}/dt) \times \mathbf{r}$ has magnitude $m\omega^2 y$, and it points in the downward vertical direction.

A person of mass m at point P therefore interprets the situation as, "I am not accelerating with respect to the cone. Therefore, the net force on me is zero. And indeed, the upward normal force F_P from the cone, with magnitude $m\omega^2 y$, is exactly balanced by the mysterious downward force F_{az}, also with magnitude $m\omega^2 y$."