FINITE GROUPS WHOSE PRIME GRAPHS ARE REGULAR

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ABSTRACT. Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G. Let $\mathrm{cd}(G)$ be the set of all character degrees of G and denote by $\rho(G)$ the set of primes which divide some character degrees of G. The prime graph $\Delta(G)$ associated to G is a graph whose vertex set is $\rho(G)$ and there is an edge between two distinct primes p and q if and only if the product pq divides some character degree of G. In this paper, we show that the prime graph $\Delta(G)$ of a finite group G is 3-regular if and only if it is a complete graph with four vertices.

1. INTRODUCTION

Given a finite group G, let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G and let $\mathrm{cd}(G) = \{\chi(1) \,|\, \chi \in \mathrm{Irr}(G)\}$ be the set of character degrees of G. The set of primes which divide some character degrees of G is denoted by $\rho(G)$. The *prime graph* $\Delta(G)$ associated to G is a graph whose vertex set is $\rho(G)$ and there is an edge between two distinct primes p and q in $\rho(G)$ if and only if the product pq divides some character degree $a \in \mathrm{cd}(G)$. The prime graph $\Delta(G)$ of a finite group G is a useful tool in studying the character degree set $\mathrm{cd}(G)$. This graph has been studied extensively over the last 20 years. We refer the readers to a recent survey by M. Lewis [9] for results concerning this graph and related topics.

In this paper, we are going to study the following question: Which graphs can occur as the prime graphs of finite groups? This is one of the basic questions in the character theory of finite groups. Although a complete answer to this question is yet to be found, many restrictions on the structure of the prime graph $\Delta(G)$ have been obtained. For example, it is known that $\Delta(G)$ has at most three connected components and if $\Delta(G)$ is connected, then its diameter is bounded above by three. (See [9, Theorems 6.4, 6.5]). For finite solvable groups, Pálfy's Condition [14] asserts that given any three distinct primes in $\rho(G)$, there is always an edge connecting two primes among those primes. This condition is very useful in determining which graphs can occur as the prime graphs of finite solvable groups. In particular, this condition implies that if G is finite solvable, then $\Delta(G)$ has at most two connected components and if $\Delta(G)$ has exactly two connected components, then each component is complete.

Unfortunately, this condition does not hold true in general. Nevertheless, it was proved in [13] that if $\pi \subseteq \rho(G)$ with $|\pi| \geq 4$, then there is an edge connecting two distinct primes in π.

The main purpose of this paper is to classify all k-regular graphs which can occur as $\Delta(G)$ for some finite group G, for $0 \leq k \leq 3$. Recall that a graph $\mathscr{G}$ is called k-regular for some integer $k \geq 0$, if every vertex of $\mathscr{G}$ has the same degree k. Combining results in [11, 15], one can easily classify all k-regular prime graphs for $0 \leq k \leq 2$. (See Proposition 2.7 in Section 2). In particular, if $\Delta(G)$ is 2-regular, then $\Delta(G)$ is a triangle or a square. For 3-regular graphs, we obtain the following result.

Theorem A. *The prime graph $\Delta(G)$ of a finite group G is 3-regular if and only if it is a complete graph with four vertices.*

Obviously, if $\Delta(G)$ is a complete graph with four vertices, then it is 3-regular. Therefore, we mainly focus on the 'only if' part. There are examples of both solvable and nonsolvable groups whose prime graphs are complete graphs with four vertices. For nonsolvable groups, we can simply take $G \cong A_7$, the alternating group of degree 7. For solvable groups, we can take G to be a direct product of two solvable groups H and K, where both $\Delta(H)$ and $\Delta(K)$ are complete graphs with two vertices and $\rho(G) \cap \rho(H)$ is empty.

We mention that an analogous result for conjugacy class sizes was obtained by Bianchi et al. in [2], where the authors proved that the common-divisor graph $\Gamma(G)$, defined on the set of non-central conjugacy class sizes of a finite group G, is 3-regular if and only if it is a complete graph with four vertices and they conjectured that $\Gamma(G)$ is a k-regular graph if and only if it is a complete graph with $k+1$ vertices. Recently, this conjecture has been proved in [1].

The paper is organized as follows. In Section 2, we obtain an upper bound for the number of vertices of the prime graph $\Delta(G)$ of a finite group G in terms of the maximal degree d and the independent number of $\Delta(G)$ under the assumption that $\Delta(G)$ contains no subgraph isomorphic to a complete graph with $d+1$ vertices. (See Corollary 2.5). This result may be useful in studying the prime graphs with bounded degrees. In Section 3, we prove Theorem A for solvable groups. This is achieved in Theorem 3.2. Section 4 is devoted to proving Theorem A for nonsolvable groups. This is the main part of the paper. Finally, in the last section, for each even integer $k \geq 2$, we construct a finite solvable group whose prime graph is k-regular with $k+2$ vertices.

All groups in this paper are assumed to be finite, all characters are complex characters and all graphs are finite, simple, undirected graphs (no loop nor multiple edge). We refer to [8] for the notation of character theory of finite groups and to [3] for terminology in graph theory. For an integer n, we write $\pi(n)$ for the set of all prime divisors of n. We write $\pi(G)$ instead of $\pi(|G|)$ for the set of all prime divisors of $|G|$.

If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. Finally, we write $\mathrm{Irr}(G|\theta)$ for the set of all irreducible constituents of θ^G.

2. Prime graphs of groups

In this section, we recall some graph theoretic terminologies and some known results in both graph and group theories which will be needed in this paper. We begin with some basic definitions and results in graph theory.

Let $\mathscr{G} = (V, E)$ be a graph of order $n = |V|$ with vertex set V and edge set E. Let v be a vertex of $\mathscr{G}$. The *degree* of v is the number of edges of $\mathscr{G}$ incident to v. A vertex $v \in V$ is said to be an *odd vertex* if its degree is odd. The following elementary result, which is a consequence of the Hand-Shaking Lemma, is well known.

Lemma 2.1. *The number of odd vertices in a graph is even.*

A graph $\mathscr{G}$ is called *k-regular* (or *regular of valency k*) for some integer $k \geq 0$, if every vertex of $\mathscr{G}$ has the same degree k. We call a 3-regular graph a *cubic graph*. From Lemma 2.1, if $\mathscr{G}$ is k-regular for some odd integer $k \geq 1$, then the order of $\mathscr{G}$ must be even since every vertex of $\mathscr{G}$ is an odd vertex. For an integer $n \geq 3$, we denote by K_n a complete graph of order n, that is, a graph with n vertices in which all pairs of distinct vertices are adjacent. A complete graph of order four is called a *complete square*. A graph $\mathscr{G}$ is said to be K_n-*free* for some $n \geq 3$ if $\mathscr{G}$ has no subgraph isomorphic to K_n. Clearly, $\mathscr{G}$ is K_3-free if and only if $\mathscr{G}$ has no triangle. Observe that a connected k-regular graph $\mathscr{G}$ for some $k \geq 2$ is K_{k+1}-free if and only if it is not K_{k+1}. A subset I of V is an *independent set* if no two of its elements are adjacent. The independent number $\alpha(\mathscr{G})$ of $\mathscr{G}$ is the maximal size of independent sets in $\mathscr{G}$. Finally, the *chromatic number* of $\mathscr{G}$, denoted by $\chi(\mathscr{G})$, is the minimal number of colors in a vertex coloring of $\mathscr{G}$.

It is well known that $\chi(\mathscr{G}) \geq n/\alpha(\mathscr{G})$. (See for instance [3, page 147]). Let d be the maximal degree of a graph $\mathscr{G}$. From the definition, we have that

$$\chi(\mathscr{G}) \leq d + 1.$$

Brooks [4] classified all graphs for which the equality $\chi(\mathscr{G}) = d+1$ holds. In particular, if $\chi(\mathscr{G}) = d + 1$, then $\mathscr{G}$ contains K_{d+1} or $d = 2$ and $\mathscr{G}$ contains an odd cycle. Using this result, we obtain the following upper bound for the order of a graph in terms of the independent number $\alpha(\mathscr{G})$ and the maximal degree d of $\mathscr{G}$.

Lemma 2.2. *Let $\mathscr{G}$ be a graph of order n with maximal degree $d \geq 3$. Suppose that $\mathscr{G}$ is K_{d+1}-free. Then $n \leq \alpha(\mathscr{G})d$.*

Proof. By Brooks' Theorem [4], we deduce that $\chi(\mathscr{G}) \leq d$. Since $\chi(\mathscr{G}) \geq n/\alpha(\mathscr{G})$, we obtain that $n/\alpha(\mathscr{G}) \leq d$ or equivalently $n \leq \alpha(\mathscr{G})d$ as wanted. $\square$

Notice that if $\mathscr{G}$ is connected with maximal degree $d \geq 3$ which is not K_{d+1}, then the order of $\mathscr{G}$ is bounded above by $\alpha(\mathscr{G})d$.

Let G be a group and let $\pi \subseteq \rho(G)$. For solvable groups, Pálfy [14] showed that there is always an edge between two primes in π whenever $|\pi| \geq 3$. For arbitrary groups, Moretó and Tiep [13] proved that a similar conclusion holds provided that $|\pi| \geq 4$. We summarise these results in the following lemma.

Lemma 2.3. *Let G be a group and let $\pi \subseteq \rho(G)$.*

(1) (Pálfy's Condition [14, Theorem]). *If G is solvable and $|\pi| \geq 3$, then there exist two distinct primes u, v in π and $\chi \in \mathrm{Irr}(G)$ such that $uv \mid \chi(1)$.*

(2) (Moretó-Tiep's Condition [13, Main Theorem]). *If $|\pi| \geq 4$, then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is divisible by two distinct primes in π.*

Translating these results into graph theoretic terminology, we obtain the following.

Lemma 2.4. *Let G be a group with prime graph $\Delta(G)$. Then $\alpha(\Delta(G)) \leq 3$. Moreover, if G is solvable, then $\alpha(\Delta(G)) \leq 2$.*

Combining the previous two lemmas, we obtain an upper bound for the order of the prime graph $\Delta(G)$ of a group G.

Corollary 2.5. *Let G be a group with prime graph $\Delta(G)$. Suppose that the maximal degree of $\Delta(G)$ is $d \geq 3$ and $\Delta(G)$ is K_{d+1}-free. Then $|\rho(G)| \leq 3d$ and if G is solvable, then $|\rho(G)| \leq 2d$. In particular, if $\Delta(G)$ is connected k-regular for some $k \geq 3$ which is not K_{k+1}, then $|\rho(G)| \leq 3k$; and if G is solvable, then $|\rho(G)| \leq 2k$.*

We now classify regular graphs with small valency which might occur as $\Delta(G)$ for some group G using results we have collected so far. We first consider the case $\Delta(G)$ is disconnected.

Lemma 2.6. *Let G be a group and let $k \geq 0$ be an integer. Suppose that $\Delta(G)$ is a disconnected k-regular graph. Then $k = 0$.*

Proof. Assume first that G is solvable. By [9, Corollary 4.2], we know that $\Delta(G)$ has exactly two connected components with vertex sets ρ_1 and ρ_2, where $n_1 = |\rho_1| \leq n_2 = |\rho_2|$ and that each connected component is complete. It follows that each vertex in ρ_1 has degree $n_1 - 1$ and each vertex in ρ_2 has degree $n_2 - 1$, respectively. Hence, $n_1 - 1 = k = n_2 - 1$, so $n_1 = n_2$. Now [9, Theorem 4.3] yields that $n_2 \geq 2^{n_1} - 1$, which forces $n_1 = n_2 = 1$ and so $k = 0$.

Assume now that G is nonsolvable. By [9, Theorem 6.4], $\Delta(G)$ has at most 3 connected components and one of which is an isolated vertex. Since $\Delta(G)$ is k-regular, we deduce that $k = 0$. The proof is now complete. $\square$

The next result gives a classification of all k-regular graphs with $0 \leq k \leq 2$, which can occur as $\Delta(G)$ for some group G.

Proposition 2.7. *Let G be a group. Suppose that the prime graph $\Delta(G)$ is k-regular for some k with $0 \leq k \leq 2$. Then the following hold.*

(1) If $k = 0$, then $\Delta(G)$ has at most 3 vertices and each vertex of $\Delta(G)$ is isolated.
(2) If $k = 1$, then $\Delta(G)$ is isomorphic to K_2, a complete graph with 2 vertices. In particular, G is solvable.
(3) If $k = 2$, then $\Delta(G)$ is either a triangle or a square. Moreover, G is solvable if $\Delta(G)$ is a square.

Proof. If $k = 0$, then the result is clear since $\Delta(G)$ has at most three connected components. Assume now that $k \geq 1$. By Lemma 2.6, $\Delta(G)$ is a connected k-regular graph. If $k = 1$, then $\Delta(G)$ is a line with two vertices and by using Ito-Michler's Theorem [12, Theorem 5.5] and Burnside's $p^a q^b$ Theorem [8, Theorem 3.10], we deduce that G is solvable. Finally, assume that $k = 2$. It follows that $\Delta(G)$ is a cycle of length $n = |\rho(G)| \geq 3$. If $n = 3$, then $\Delta(G)$ is a triangle and G could be solvable or nonsolvable. Assume now that $n > 3$. It follows from [15, Theorem C] that $n = 4$ and so $\Delta(G)$ is a square. Now [11, Theorem B] yields that G is solvable. $\square$

In the last result of this section, we eliminate all but four cubic graphs of order at least 6 which might occur as the prime graph of some group. These graphs will be ruled out in the next two sections.

Proposition 2.8. *If the prime graph $\Delta(G)$ of a group G is a connected 3-regular graph with $|\rho(G)| \geq 6$, then $\Delta(G)$ is isomorphic to one of the graphs in Figures 1-4.*
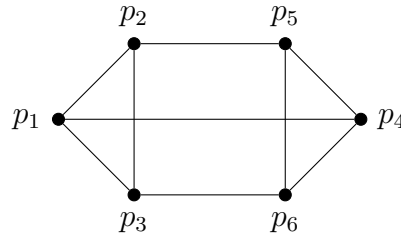

FIGURE 1. A cubic graph of order six

Proof. Suppose that $\Delta(G)$ is a connected 3-regular graph with $|\rho(G)| \geq 6$ for some group G. Since $\Delta(G)$ is 3-regular, every vertex of $\Delta(G)$ is an odd vertex and thus by Lemma 2.1, $|\rho(G)|$ must be even. By Corollary 2.5, $|\rho(G)| \leq 9$ since it is connected but it is not isomorphic to K_4. Therefore, $|\rho(G)| = 6$ or 8. Writing $\rho(G) = \{p_i\}_{i=1}^n$ with $n = |\rho(G)|$. By [15, Theorem A], we know that $\Delta(G)$ always contains a triangle with vertex set, say $\{p_1, p_2, p_3\}$. We consider the following cases.

Case 1. $\Delta(G)$ has a vertex which is adjacent to two distinct vertices in $\{p_i\}_{i=1}^3$.

Without loss of generality, assume that $p_4 \in \rho(G)$ is adjacent to two distinct vertices in $\{p_i\}_{i=1}^3$, say p_1 and p_2. Since p_1 and p_2 are joined to each other and to
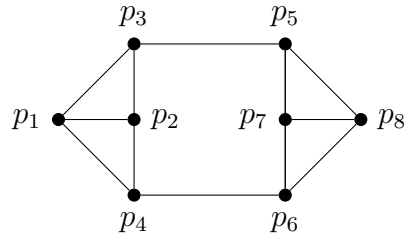
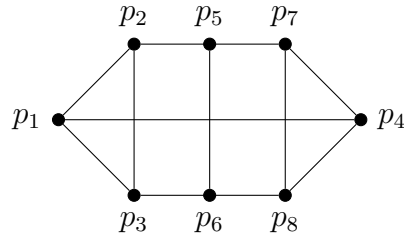FIGURE 2. A cubic graph of order eight with four triangles

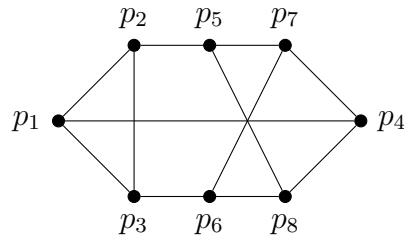

FIGURE 3. A cubic graph of order eight with two triangles

FIGURE 4. A cubic graph of order eight with one triangle

both p_3 and p_4, they are not adjacent to any other vertices. As both p_3 and p_4 have degree 3 in $\Delta(G)$, one of the following cases holds.

(i) Both p_3 and p_4 are adjacent to the same vertex, say p_5.

Assume that $|\rho(G)| = 6$. As $\Delta(G)$ is connected, p_6 must be adjacent to p_5 but it cannot be adjacent to any other vertices since otherwise $\Delta(G)$ would have a vertex of degree at least four, which is impossible. However, this case cannot happen since we cannot add more edges to this graph to obtain a cubic graph of order six.

Now assume that $|\rho(G)| = 8$. Let $\tau = \{p_6, p_7, p_8\}$. As both p_3 and p_4 are adjacent to p_1, p_2 and p_5, they are not joined to each other and to any vertices in τ. So, for

any $u \neq v \in \tau$, by applying Moretó-Tiep's Condition for $\{u, v, p_3, p_4\}$, we see that u and v must be joined to each other. In particular, τ is a vertex set of a triangle in $\Delta(G)$. As $\Delta(G)$ is connected, p_5 must be adjacent to some vertex in τ, say p_6. But then we cannot add more edges to this graph to obtain a cubic graph since p_7 and p_8 cannot be joined to any other vertices. Hence, this case cannot happen.

(ii) p_3 and p_4 are adjacent to two distinct vertices, say p_5 and p_6, respectively.

If $|\rho(G)| = 6$, then we cannot add more edges to this graph to obtain a cubic graph. Hence, this case cannot occur.

Therefore, we can assume that $|\rho(G)| = 8$. Let $\tau = \{p_7, p_8\}$. Notice that p_3 and p_4 are not adjacent in $\Delta(G)$. By Moretó-Tiep's Condition for $\{p_3, p_4, p_7, p_8\}$, p_7 and p_8 are joined to each other. Assume that p_5 and p_6 are adjacent. Since $\Delta(G)$ is connected and each $p_i, 1 \leq i \leq 4$, cannot be adjacent to any other vertices, p_5 must be adjacent to either p_7 or p_8, say p_7. As the degree of p_6 is three, it must be adjacent to either p_7 or p_8. However, both cases are impossible as we cannot add more edges to this graph to obtain a cubic graph. Thus, we can assume that p_5 and p_6 are not joined to each other. Hence, each of them must be adjacent to both p_7 and p_8 and thus $\Delta(G)$ is isomorphic to the graph in Fig. 2.

Case 2. No vertex in $\rho(G)$ is adjacent to two distinct vertices in $\{p_i\}_{i=1}^3$.

As each $p_i, i = 1, 2, 3$, has degree three in $\Delta(G)$, there exist $\{p_j\}_{j=4}^6 \subseteq \rho(G) - \{p_i\}_{i=1}^3$ such that each p_k is adjacent to p_{k+3} for $k = 1, 2, 3$.

Assume first that $|\rho(G)| = 6$. Then $\rho(G) = \{p_i\}_{i=1}^6$. Clearly, $\Delta(G)$ is a 3-regular graph if and only if $\{p_i\}_{i=4}^6$ forms a triangle. Hence, $\Delta(G)$ is the graph in Fig. 1. Assume next that $|\rho(G)| = 8$. Let $\tau = \{p_7, p_8\}$. Then the following cases hold.

(i) Assume that some vertex p_j with $4 \leq j \leq 6$ is adjacent to the remaining vertices in $\{p_i\}_{i=4}^6$. Without loss of generality, assume that p_6 is adjacent to p_4 and p_5. Since $\Delta(G)$ is connected and $|\rho(G)| = 8$, p_4 and p_5 are not adjacent in $\Delta(G)$. By Moretó-Tiep's Condition for $\{p_1, p_6, p_7, p_8\}$, p_7 and p_8 are adjacent in $\Delta(G)$. As p_4 has degree three, it must be adjacent to p_7 or p_8, say p_7. Similarly, p_5 is adjacent to either p_7 or p_8. However, both cases are impossible as we cannot add more edges to this graph to obtain a cubic graph.

(ii) Assume next that there is exactly one edge among vertices $\{p_i\}_{i=4}^6$, say $\{p_5, p_6\}$. Then p_4 is not adjacent to p_5, p_6. Since p_4 has degree 3 in $\Delta(G)$ and it is not adjacent to any vertices in $\{p_i\}_{i=2}^6$, it is adjacent to both p_7 and p_8. Similarly, as p_5 has degree 3 in $\Delta(G)$, it is adjacent to either p_7 or p_8, say p_7. It follows that p_6 is adjacent to either p_7 or p_8. If p_6 is adjacent to p_7, then p_8 is not adjacent to any other vertices, except for p_4, which is impossible. Thus p_6 is adjacent to p_8. By joining p_7 and p_8, we obtain the graph in Fig. 3.

(iii) Assume that there is no edge among vertices $p_j, 4 \leq j \leq 6$. For each $i \in \{1, 2, 3\}$, p_i is joined to p_{i+3} and to the remaining vertices in $\{p_\ell\}_{\ell=1}^3$, so it is not adjacent to any other vertices. Hence, each vertex $p_j, 4 \leq j \leq 6$, is adjacent to

both vertices p_7 and p_8. Thus, we obtain the graph in Fig. 4. The proof is now complete. □

3. Prime graphs of solvable groups

In this section, we prove Theorem A for the solvable groups. We first eliminate the graph in Fig. 1 from being the prime graph of some solvable group.

Lemma 3.1. *Suppose that the prime graph $\Delta(G)$ of a group G is isomorphic to the graph in Fig. 1. Then $G' = G''$. In particular, G is nonsolvable.*

Proof. By way of contradiction, assume that $G' \neq G''$. Then G/G'' is a nonabelian solvable group. Let N be a maximal normal subgroup of G such that G/N is a minimal nonabelian solvable group. By [8, Lemma 12.3], the following cases hold.

Case 1. G/N is a nonabelian r-group for some prime r.

In this case, G/N has an irreducible character $\tau \in \operatorname{Irr}(G/N)$ of degree r^a for some prime r and some integer $a \geq 1$. We now claim that r is adjacent to every prime in $\rho(G) - \{r\}$. Indeed, for every prime $p \in \rho(G) - \{r\}$, there exists $\chi \in \operatorname{Irr}(G)$ with $p \mid \chi(1)$. If $r \mid \chi(1)$, then r and p are joined to each other and we are done. So, we can assume that $r \nmid \chi(1)$. It follows that $\gcd(\chi(1), |G : N|) = 1$, hence $\chi_N \in \operatorname{Irr}(G)$. By Gallagher's Theorem [8, Corollary 6.17], $\chi\tau \in \operatorname{Irr}(G)$, so $pr \mid \chi(1)\tau(1)$, therefore, p and r are adjacent in $\Delta(G)$. Thus, r is adjacent to every prime in $\rho(G) - \{r\}$. Since $|\rho(G)| = 6$, r has degree five in $\Delta(G)$, which is a contradiction.

Case 2. G/N is a Frobenius group with Frobenius kernel F/N, an elementary abelian r-group for some prime r, and $f = |G : F| \in \operatorname{cd}(G)$ with $\gcd(r, f) = 1$.

By [8, Theorem 12.4], we know that for every $\psi \in \operatorname{Irr}(F)$, either $f\psi(1) \in \operatorname{cd}(G)$ or $r \mid \psi(1)$. Since $\Delta(G)$ has no complete subgraph isomorphic to K_4, we deduce that $|\pi(\chi(1))| \leq 3$ for all $\chi \in \operatorname{Irr}(G)$. Notice that $\Delta(G)$ has exactly two triangles with vertex sets $\{p_1, p_2, p_3\}$ and $\{p_4, p_5, p_6\}$. The remaining edges of $\Delta(G)$ are $\{p_i, p_{i+3}\}$ for $1 \leq i \leq 3$. As $f \in \operatorname{cd}(G)$, we have that $|\pi(f)| \leq 3$. Hence, $|\pi(f) \cup \{r\}| \leq 4$, so there exists $j \in \{1, 2, 3\}$ such that $r \notin \{p_j, p_{j+3}\}$ and $\pi(f) \nsubseteq \{p_j, p_{j+3}\}$. Thus, we can find $s \in \pi(f)$ with $s \notin \{p_j, p_{j+3}\}$. Let $\chi \in \operatorname{Irr}(G)$ such that $p_j p_{j+3} \mid \chi(1)$. As $\Delta(G)$ contains only two triangles with vertex sets $\{p_1, p_2, p_3\}$ and $\{p_4, p_5, p_6\}$, respectively, we deduce that $\pi(\chi(1)) = \{p_j, p_{j+3}\}$. Let $\theta \in \operatorname{Irr}(F)$ be an irreducible constituent of χ_F. Since $\theta(1) \mid \chi(1)$, we deduce that $r \nmid \theta(1)$, hence $f\theta(1) \in \operatorname{cd}(G)$. Writing $\chi(1) = k\theta(1)$. Then $k \mid \gcd(\chi(1), f)$, so $s \nmid k$, hence $s \mid f/k$. We now have that $f\theta(1) = f\chi(1)/k = \chi(1)(f/k)$ is divisible by s, p_j and p_{j+3}, so the subgraph on $\{s, p_j, p_{j+3}\}$ of $\Delta(G)$ is a triangle, a contradiction. Therefore, $G' = G''$ as wanted.

Finally, if G is solvable, then $G' = G'' = 1$ so G is abelian, which is impossible since $|\rho(G)| = 6$. The proof is now complete. □

We now prove the main result of this section.

Theorem 3.2. *Let G be a solvable group. If $\Delta(G)$ is a cubic graph, then $\Delta(G)$ is isomorphic to a complete graph of order four.*

Proof. Suppose that $\Delta(G)$ is a cubic graph for some solvable group G. Then $|\rho(G)| \geq 4$ and every vertex of $\Delta(G)$ has the same degree 3. By Lemma 2.6, $\Delta(G)$ is connected. As every vertex of $\Delta(G)$ is an odd vertex, Lemma 2.1 implies that $|\rho(G)|$ is even. If $|\rho(G)| = 4$, then $\Delta(G)$ is a complete square and we are done. So, assume that $|\rho(G)| \geq 6$. By Corollary 2.5, we have that $|\rho(G)| \leq 6$, so $|\rho(G)| = 6$. By Proposition 2.8, $\Delta(G)$ is isomorphic to the graph in Fig. 1. Now Lemma 3.1 yields a contradiction. Thus $\Delta(G)$ must be a complete square. $\square$

4. PRIME GRAPHS OF NONSOLVABLE GROUPS

In this section, we give a proof of Theorem A for nonsolvable groups. By the Ito-Michler theorem [12, Theorem 5.5], we know that $\rho(G) = \pi(G)$ for any almost simple groups G since G has no nontrivial normal abelian Sylow subgroup. This fact will be used without further reference. We first classify all simple groups whose prime graphs are K_4-free.

Lemma 4.1. *Let S be a nonabelian simple group. Suppose that the prime graph $\Delta(S)$ is K_4-free. Then one of the following cases holds.*

(1) $S \cong \mathrm{M}_{11}$ *or* J_1;
(2) $S \cong \mathrm{A}_n$ *with* $n \in \{5, 6, 8\}$;
(3) $S \cong \mathrm{PSL}_2(q)$ *with* $q = p^f \geq 4$ *and* $|\pi(q \pm 1)| \leq 3$, *where p is prime;*
(4) $S \cong \mathrm{PSL}_3(q)$ *with* $q \in \{3, 4, 8\}$;
(5) $S \cong \mathrm{PSU}_3(q)$ *with* $q \in \{3, 4, 9\}$;
(6) $S \cong \mathrm{PSp}_4(3)$ *or* ${}^2\mathrm{B}_2(q^2)$ *with* $q^2 = 2^3$ *or* 2^5.

Proof. If $|\pi(S)| = 3$, then S is isomorphic to one of the following simple groups

$$\mathrm{A}_5, \mathrm{A}_6, \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2), \mathrm{PSL}_2(7), \mathrm{PSL}_2(8), \mathrm{PSU}_3(3), \mathrm{PSL}_3(3), \mathrm{PSL}_2(17)$$

by [7, Table 1]. Clearly, these groups are K_4-free and they appear somewhere in the conclusion of the lemma. Hence, we can assume that $|\pi(S)| \geq 4$.

If S is a sporadic simple group or an alternating group, then by using Theorems 2.1 and 3.1 in [16], we can easily deduce that $\Delta(S)$ is K_4-free if and only if S is one of the groups in (1) and (2).

Assume that $S \cong {}^2\mathrm{B}_2(q^2)$ with $q^2 = 2^{2m+1}$ and $m \geq 1$. By [16, Theorem 4.1], the subgraph of $\Delta(S)$ on $\pi((q^2 - 1)(q^4 + 1))$ is complete. Since $\Delta(S)$ is K_4-free, $|\pi((q^2 - 1)(q^4 + 1))| \leq 3$ and thus $3 \leq |\pi(S)| \leq 4$ as $\pi(S) = \{2\} \cup \pi((q^2 - 1)(q^4 + 1))$. Hence, $|\pi(S)| = 4$ and so by [7, Table 2], $m = 1$ or 2. These cases appear in (6).

If $S \cong \mathrm{PSL}_2(q)$, then the result is clear as the character degree set of S is known. In particular, as $q \pm 1 \in \mathrm{cd}(G)$, $|\pi(q \pm 1)| \leq 3$. This gives (3).

Assume that $S \cong \mathrm{PSL}_3(q)$ or $\mathrm{PSU}_3(q)$ ($q > 2$). If $S \cong \mathrm{PSL}_3(4)$ or $\mathrm{PSL}_4(2) \cong \mathrm{A}_8$, then S is K_4-free. Assume next that S is not one of these groups. By Theorems 5.3

and 5.5 in [16], the subgraph of $\Delta(S)$ on $\pi(S) - \{p\}$ is complete, where $q = p^f$ and p is a prime. Obviously, $|\pi(S)| = 4$ and thus (4) and (5) hold by using [7, Table 2].

For the remaining simple groups, $\Delta(S)$ is complete by using [16] again and thus $\Delta(S)$ contains a subgraph isomorphic to K_4 since $|\pi(S)| \geq 4$. Hence, $\Delta(S)$ is not K_4-free in any of these cases. This completes the proof. □

We now deduce the following result for almost simple groups.

Lemma 4.2. *Let S be a nonabelian simple group and let G be an almost simple group with $S \trianglelefteq G \leq \operatorname{Aut}(S)$. Suppose that $\Delta(G)$ is K_4-free with maximal degree $d \leq 3$. Then S is one of the following simple groups:*

(1) *$S \cong \mathrm{M}_{11}$ and $|\pi(S)| = 4$;*
(2) *$S \cong \mathrm{A}_n$ with $n \in \{5, 6, 8\}$ and $|\pi(S)| \leq 4$;*
(3) *$S \cong \mathrm{PSL}_2(q)$ with $q = p^f \geq 4$ and $|\pi(q \pm 1)| \leq 3$, where p is prime; Moreover, if q is odd, then $|\pi(q^2 - 1)| \leq 4$ so $|\pi(S)| \leq 5$; and if q is even, then $|\pi(S)| \leq 7$;*
(4) *$S \cong \mathrm{PSL}_3(q)$ with $q \in \{2, 3, 4, 8\}$ and $|\pi(S)| \leq 4$;*
(5) *$S \cong \mathrm{PSU}_3(q)$ with $q \in \{3, 4, 9\}$ and $|\pi(S)| \leq 4$;*
(6) *$S \cong \mathrm{PSp}_4(3)$ or ${}^2\mathrm{B}_2(q^2)$ with $q^2 = 2^3$ or 2^5 and $\pi(S) \leq 4$;*

and one of the following cases holds.

(i) *$\pi(G) = \pi(S)$ and either $3 \leq |\pi(S)| \leq 4$ or $S \cong \mathrm{PSL}_2(q)$ with $|\pi(q^2 - 1)| \geq 4$, q being a prime power and $|\pi(S)| \leq 7$ or*
(ii) *$\pi(G) = \pi(S) \cup \{r\}$, $S \cong \mathrm{PSL}_2(q)$, q being a prime power, $r \in \pi(G) - \pi(S)$ is adjacent to every prime in $\pi(q^2 - 1)$, and $|\pi(G)| = |\pi(S)| + 1 = 5$.*

In all cases, we have $|\pi(G)| \leq 7$. Moreover, if $|\pi(G)| \geq 6$, then $G = S$, where $S \cong \mathrm{PSL}_2(2^f)$ with $f \geq 10$; and if $|\pi(G)| \geq 5$, then $S \cong \mathrm{PSL}_2(q)$ for some prime power $q \geq 11$.

Proof. As $S \trianglelefteq G$, the graph $\Delta(S)$ is a subgraph of $\Delta(G)$ so $\Delta(S)$ is K_4-free and all vertices of $\Delta(S)$ have degree at most 3. Now the possibilities for S are obtained from Lemma 4.1 by excluding simple groups having a vertex with degree exceeding 3. Now (i) is obvious. For (ii), let $\tau = \pi(G) - \pi(S)$ and assume that τ is nonempty. Clearly, this implies that either $S \cong {}^2\mathrm{B}_2(8)$ or $S \cong \mathrm{PSL}_2(q)$ for some prime power $q \geq 4$. If the first case holds, then $G = {}^2\mathrm{B}_2(8) \cdot 3$; however $\Delta(G)$ is not K_4-free by using [5]. So, assume that the latter case holds. Then $|\pi(S)| \geq 4$ and hence $|\pi(q^2 - 1)| \geq 3$. Observe that all primes in τ are odd and if m is the product of all distinct primes in τ, then by [17, Theorem A], we deduce that $m(q \pm 1)$ divide some character degrees of G. As $|\pi(S)| \geq 4$, we have that $|\pi(q + \delta)| \geq 2$ for some $\delta \in \{\pm 1\}$ so that $|\pi(m)| = 1$ since otherwise $m(q + \delta)$ would have four distinct prime divisors, a contradiction. Thus $\pi(m) = \pi(G) - \pi(S) = \{r\}$ and r is adjacent to all primes in $\pi(q^2 - 1)$ as wanted. Furthermore, as r has degree at most 3 in $\Delta(G)$, we have that $|\pi(q^2 - 1)| = 3$, hence $|\pi(S)| = 4$ and $|\pi(G)| = 5$.

Clearly, $|\pi(G)| \leq 7$ by (i) and (ii). Now assume that $|\pi(G)| \geq 6$. By (ii), we have that $\pi(G) = \pi(S)$. In particular, $|\pi(S)| \geq 6$. Hence, $S \cong \mathrm{PSL}_2(2^f)$ with $f \geq 10$. We next claim that $G = S$. Suppose by contradiction that $G \neq S$. By [17, Theorem A] again, we know that $|G : S|(2^f \pm 1) \in \mathrm{cd}(G)$. Let u be a prime divisor of $|G : S|$. Then u is adjacent to every prime in $\pi(2^{2f} - 1) - \{u\}$. As $|\pi(S)| \geq 6$, we have that $|\pi(2^{2f} - 1)| \geq 5$ and thus the degree of u in $\Delta(G)$ is at least four, contradicting the hypothesis of the lemma. The last claim is clear. The proof is now complete. $\square$

We will need several results before we can prove the main result of this section. In the next two lemmas, we obtain some restrictions on the structure of nonsolvable groups whose prime graph is a subgraph of a 3-regular graph with at least five vertices.

Lemma 4.3. *Suppose that the prime graph $\Delta(G)$ of some nonsolvable group G is K_4-free with maximal degree $d \leq 3$ and $|\rho(G)| \geq 5$. Then every nonabelian chief factor of G is simple.*

Proof. Assume that M/N is a nonabelian chief factor of G. Then $M/N \cong S^k$ for some nonabelian simple group S and some integer $k \geq 1$. Let $C/N = \mathrm{C}_{G/N}(M/N)$. Then $C \trianglelefteq G$ and G/C has a unique minimal normal subgroup M/N. Assume that $k > 1$. Since G/C has no nontrivial normal abelian Sylow subgroup, Ito-Michler's theorem yields that $\rho(G/C) = \pi(G/C)$ and thus $|\rho(G/C)| = |\pi(G/C)| \geq 3$ by [8, Theorem 3.10]. Applying [10, Main Theorem], the prime graph $\Delta(G/C)$ is complete. Since $\Delta(G/C)$ is a subgraph of $\Delta(G)$ which is K_4-free, $\Delta(G/C)$ is also K_4-free and thus $\Delta(G/C)$ must be a triangle with vertex set $\{p_i\}_{i=1}^3$. Let L be a normal subgroup of MC such that $L/C \cong S$. Since $\rho(G/C) = \pi(G/C)$, for any $r \in \pi := \rho(G) - \pi(G/C)$, we deduce that $r \in \rho(C)$ and thus there exists $\theta \in \mathrm{Irr}(C)$ with $r \mid \theta(1)$. By [15, Lemma 4.2], either θ extends to $\theta_0 \in \mathrm{Irr}(L)$ or $\psi(1)/\theta(1)$ is divisible by two distinct primes in $\pi(L/C)$ for some $\psi \in \mathrm{Irr}(L|\theta)$. If the first case holds, then r is adjacent to every prime $p_i, 1 \leq i \leq 3$, and so the subgraph of $\Delta(G)$ on $\{r, p_1, p_2, p_3\}$ is a complete square, which is impossible. If the latter case holds, then r is adjacent to two distinct primes, say $p_i \neq p_j$. As $|\rho(G)| \geq 5$, we can find $r \neq s \in \pi$, and thus with the same argument as above, s is also adjacent to two distinct primes in $\{p_k\}_{k=1}^3$. It follows that there exists a prime $p_m, 1 \leq m \leq 3$, such that p_m is adjacent to both r and s in $\Delta(L)$. As $\Delta(G/C)$ is a triangle, p_m is adjacent to every prime in $\pi(G/C) - \{p_m\}$, so its degree in $\Delta(G)$ is at least 4, a contradiction. Therefore, $k = 1$ as wanted. $\square$

Lemma 4.4. *Assume the hypotheses of Lemma 4.3. Let N be the solvable radical of G and let M/N be a chief factor of G. Then G/N is almost simple with socle M/N.*

Proof. Since N is the largest normal solvable subgroup of G, M/N is nonabelian and so by Lemma 4.3, $M/N \cong S$, where S is a nonabelian simple group. Let $C/N = \mathrm{C}_{G/N}(M/N)$. It suffices to show that $C = N$. Suppose by contradiction that $C \neq N$ and let K/N be a minimal normal subgroup of G/N with $K \leq C$. Then K/N is a nonabelian chief factor of G, so by Lemma 4.3 again, $K/N \cong T$, where T is a

nonabelian simple group. Notice that $K \cap M = N$, $CM/N \cong C/N \times M/N \trianglelefteq G/N$ is a direct product and $\pi(T) \subseteq \pi(C/N)$, hence $2 \in \rho(C/N) \cap \pi(M/N)$. It follows that 2 is adjacent to every prime in $\rho(C/N) \cup \pi(M/N)$. As the degree of 2 in $\Delta(G)$ is at most 3, we deduce that $3 \leq |\rho(C/N) \cup \pi(M/N)| \leq 4$.

Assume first that $|\rho(C/N) \cup \pi(M/N)| = 4$. Since $|\rho(C/N)| \geq |\pi(T)| \geq 3$ and $|\pi(M/N)| \geq 3$, we deduce that $|\rho(C/N) \cap \pi(M/N)| \geq 2$. Writing

$$\pi := \rho(C/N) \cap \pi(M/N) = \{r_i\}_{i=1}^k,$$

where $r_i, 1 \leq i \leq k$, are distinct primes and $k \geq 2$. As the subgraph of $\Delta(G)$ on π is complete, we deduce that $2 \leq k \leq 3$. If $k = 2$, then $\rho(C/N) = \pi \cup \{p\}$ and $\pi(M/N) = \pi \cup \{q\}$, where p and q are distinct primes and different from $r_i, 1 \leq i \leq k$. Now, in the subgraph $\Delta(C/N \times M/N)$ of $\Delta(G)$, we see that p is adjacent to every prime in $\pi(M/N)$ and q is adjacent to every prime in $\rho(C/N)$, so $\Delta(C/N \times M/N)$ is isomorphic to K_4, a contradiction. Similarly, if $k = 3$, then the subgraph of $\Delta(G)$ on the set $\{r_i\}_{i=1}^3$ is a triangle and the vertex $p \in (\pi(M/N) \cup \rho(C/N)) - \pi$ is adjacent to every prime in π, hence the subgraph of $\Delta(G)$ on $\{p, r_1, r_2, r_3\}$ is a complete square, a contradiction.

Assume next that $|\rho(C/N) \cup \pi(M/N)| = 3$. It follows that $|\pi(M/N)| = 3$ and thus by applying [7, Table 1], we deduce that $\pi(G/C) = \pi(MC/C) = \pi(M/N)$, where G/C is an almost simple group with socle M/N. As in the previous case, we have that $|\rho(C/N)| \geq 3$ and thus

$$\rho(C/N) = \pi(M/N) = \pi(G/N) = \{p_i\}_{i=1}^3.$$

Clearly, the subgraph of $\Delta(G)$ on $\{p_i\}_{i=1}^3$ is a triangle. Since $|\rho(G)| \geq 5$ and $|\pi(G/N)| = 3$, we deduce that $\rho(N)$ contains at least two distinct primes $r_i, i = 1, 2$, such that $r_i \notin \rho(G/C) = \pi(M/N)$ for $i = 1, 2$. Let $\theta_i \in \mathrm{Irr}(N)$ with $r_i \mid \theta_i(1)$ for $i = 1, 2$. By [15, Lemma 4.2], for each $i = 1, 2$, either θ_i extends to M or $\psi_i(1)/\theta_i(1)$ is divisible by two distinct primes in $\pi(M/N)$ for some $\psi_i \in \mathrm{Irr}(M|\theta_i)$. If θ_j is extendible to M for some $j = 1, 2$, then r_j is adjacent to every prime $p_i, 1 \leq i \leq 3$, and so the subgraph of $\Delta(G)$ on $\{r_j, p_1, p_2, p_3\}$ is a complete square, a contradiction. Hence, for each $j = 1, 2$, r_j is adjacent to two distinct primes in $\{p_i\}_{i=1}^3$. It follows that some vertex $p_m, 1 \leq m \leq 3$, has degree at least four in $\Delta(G)$, which is impossible.

Therefore, $C = N$ and thus G/N is almost simple with socle M/N as wanted. $\square$

We eliminate the graphs in Figures 2-4 from being the prime graphs of any non-solvable groups in the next lemma.

Lemma 4.5. *Suppose that the prime graph $\Delta(G)$ of a nonsolvable group G is a connected 3-regular graph with $|\rho(G)| \geq 6$. Then $|\rho(G)| = 6$ and $\Delta(G)$ is isomorphic to the graph in Fig. 1.*

Proof. By Proposition 2.8, $\Delta(G)$ is isomorphic to one of the graphs in Figures 1-4 and $|\rho(G)| = 6$ or 8. As the graph in Fig. 1 is the only graph of order 6, it suffices

to show that $|\rho(G)| = 6$. By way of contradiction, assume that $|\rho(G)| > 6$. Then $|\rho(G)| = 8$ and so $\Delta(G)$ is one of the graphs in Figures 2-4. As $\Delta(G)$ is K_4-free with maximal degree $d \leq 3$. By Lemma 4.4, G/N is almost simple with nonabelian simple socle M/N, hence $\rho(G/N) = \pi(G/N)$. Let $\tau = \rho(G) - \pi(G/N)$. Then $\tau \subseteq \rho(N)$. By Lemma 4.2, $|\pi(G/N)| \leq 7$, so $|\tau| = |\rho(G)| - |\pi(G/N)| \geq 1$.

Claim 1. $1 \leq |\tau| \leq 2$ and the subgraph of $\Delta(G)$ on τ has no edge.

If $|\tau| \geq 3$, then since $\tau \subseteq \rho(N)$ with N being solvable, there is an edge among vertices in τ by Pálfy's Condition. Thus it suffices to show that there is no edge among vertices in τ. By way of contradiction, assume that $r \neq s \in \tau$ are joined to each other in $\Delta(G)$ via $\psi \in \mathrm{Irr}(G)$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of ψ_N. Since $rs \mid \psi(1)$ and $\psi(1)/\theta(1) \mid |G : N|$, where $\gcd(rs, |G : N|) = 1$, we deduce that $rs \mid \theta(1)$. By [15, Lemma 4.2], either $\chi(1)/\theta(1)$ is divisible by two distinct primes in $\pi(M/N)$ for some $\chi \in \mathrm{Irr}(M|\theta)$ or θ extends to $\theta_0 \in \mathrm{Irr}(M)$. The first case clearly cannot occur since otherwise $\chi(1)$ would be divisible by at least 4 distinct primes so $\Delta(M)$ would contain a complete square, a contradiction. Thus the latter case holds. By Gallagher's Theorem [8, Corollary 6.17], $\theta\lambda \in \mathrm{Irr}(M)$ for all $\lambda \in \mathrm{Irr}(M/N)$, hence r is adjacent to every prime in $\pi(M/N)$, where $|\pi(M/N)| \geq 3$. Since r is also adjacent to $s \notin \pi(M/N)$, its degree in $\Delta(G)$ is at least 4, a contradiction.

Claim 2. $G = M$ and $G/N \cong \mathrm{PSL}_2(2^f)$ with $f \geq 10$ and $2 \leq |\pi(2^f \pm 1)| \leq 3$.

Since $|\tau| \leq 2$ by Claim 1, we have $|\pi(G/N)| = |\rho(G)| - |\tau| \geq 6$. Lemma 4.2 now yields that $G/N = M/N \cong \mathrm{PSL}_2(2^f)$ with $f \geq 10$ and $|\pi(2^f \pm 1)| \leq 3$. It follows that $|\pi(2^f \pm 1)| \geq 2$ as $|\pi(2^{2f} - 1)| = |\pi(G/N) - \{2\}| \geq 5$.

Claim 3. For each $r \in \tau$, there exist three distinct vertices $p_k \in \pi(G/N), k = 1, 2, 3$, such that the subgraphs of $\Delta(G)$ on $\{r, p_i, p_u\}$ and $\{r, p_i, p_v\}$ are triangles, where $\{p_i, p_u, p_v\} = \{p_k\}_{k=1}^3$. Hence, $\Delta(G)$ is isomorphic to the graph in Fig. 2.

Let $r \in \tau$. Then $r \mid \mu(1)$ for some $\mu \in \mathrm{Irr}(N)$. By [15, Lemma 4.2], r together with two distinct vertices in $\pi(G/N)$, say p_1, p_2, will form a triangle in $\Delta(G)$. Since r is not adjacent to any primes in $\tau - \{r\}$ by Claim 1, r is adjacent to some prime in $\pi(G/N) - \{p_1, p_2\}$, say p_3, via $\chi \in \mathrm{Irr}(G)$.

Suppose that p_3 is not adjacent to p_1 nor p_2. Since r is of degree 3 in $\Delta(G)$ and it is adjacent to every vertex $p_i, 1 \leq i \leq 3$, it is not joined to any other vertices. In particular, $\{r, p_3\}$ is not an edge of any triangle in $\Delta(G)$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N. Then $\chi(1)/\theta(1)$ divides $|G/N|$ and thus $\chi(1)/\theta(1)$ is prime to r, therefore, $r \mid \theta(1)$. As $\{r, p_3\} \subseteq \pi(\chi(1))$, by our assumption on r and p_3 we deduce that $\pi(\chi(1)) = \{r, p_3\}$. As $|\pi(G/N)| \geq 6$, θ is not extendible to G as otherwise, r would be adjacent to every prime in $\pi(G/N)$ and so its degree in $\Delta(G)$ is at least six, a contradiction. It follows that $\chi(1)/\theta(1) = p_3^a$ for some integer $a \geq 1$. Also, as the Schur multiplier of $G/N \cong \mathrm{PSL}_2(2^f)$ with $f \geq 10$ is trivial, θ is not G-invariant. Therefore, $I = I_G(\theta) \lneq G$. By Clifford's Theory, there exists $\phi \in \mathrm{Irr}(I|\theta)$ such that $\chi = \phi^G$ and $\phi_N = e\theta$ for some integer $e \geq 1$. Then $\chi(1) = |G : I|\phi(1) = |G : I|e\theta(1)$.

Hence, $\chi(1)/\theta(1) = e|G : I| = p_3^a$. In particular, $|G : I|$ is a prime power. Also, $|G : I|$ is divisible by the index of some maximal subgroup of $G/N \cong \mathrm{PSL}_2(2^f)$. By [6, Hauptsatz II.8.27], the indices of maximal subgroups of $\mathrm{PSL}_2(2^f)$ are

$$(1) \qquad 2^{f-1}(2^f + 1), 2^{f-1}(2^f - 1), 2^f + 1, \frac{2^f(2^{2f} - 1)}{2^b(2^{2b} - 1)},$$

where $f/b = n \geq 2$ is a prime. However, since $|\pi(2^f \pm 1)| \geq 2$, none of these indices is a prime power. This contradiction shows that p_3 must be adjacent to p_1 or p_2, which proves the first part of the claim. As the graphs in Figures 3 and 4 do not contain any two triangles sharing a common edge, $\Delta(G)$ must be the graph in Fig. 2.

The final contradiction.

By Claims 2 and 3, $\Delta(G)$ is isomorphic to the graph in Figure 2, $|\rho(G)| = 8$ and $G/N \cong \mathrm{PSL}_2(2^f)$ with $f \geq 10$. Let $r \in \tau$. By Claim 3, we can assume that the subgraphs on $\{r, p_2, p_3\}$ and $\{r, p_2, p_1\}$ are two triangles in $\Delta(G)$. It follows that the subgraph of $\Delta(G)$ on $\rho(G) - \{r\}$ has exactly two triangles sharing a common edge. Hence, $|\pi(G/N)| \leq 6$ as if $|\pi(G/N)| \geq 7$, then $|\pi(G/N)| = 7$ and $\Delta(G/N)$ has two disjoint triangles, which is impossible as $\pi(G/N) \subseteq \rho(G) - \{r\}$. As $|\tau| \leq 2$ by Claim 1, we deduce that $|\pi(G/N)| = 6$ and so $\tau = \{r, s\}$ with $r \neq s$. By Claim 1, r and s are not adjacent and by Claim 3, s is a common vertex of two triangles. It follows that the subgraph of $\Delta(G)$ on $\pi(G/N)$ has no triangle with 6 vertices, hence $\Delta(G/N)$ has no triangle with 6 vertices, contradicting [15, Theorem A]. The proof is now complete. $\square$

We now complete the proof of Theorem A for nonsolvable groups.

Theorem 4.6. *Let G be a nonsolvable group. If $\Delta(G)$ is a cubic graph, then $\Delta(G)$ is isomorphic to a complete graph of order four.*

Proof. Suppose that G is a nonsolvable group such that $\Delta(G)$ is 3-regular. Then $|\rho(G)| \geq 4$ and by Lemma 2.1, $|\rho(G)|$ is even. By Lemma 2.6, $\Delta(G)$ is connected. If $|\rho(G)| = 4$, then we are done. So, we assume that $|\rho(G)| \geq 6$. By Lemma 4.5, $|\rho(G)| = 6$ and $\Delta(G)$ is the graph in Fig. 1. Let N be the solvable radical of G and let M be a normal subgroup of G such that M/N is a chief factor of G. Lemma 4.4 implies that G/N is almost simple with simple socle M/N. Let $\tau = \rho(G) - \pi(G/N)$. Then $\tau \subseteq \rho(N)$.

Claim 1. The subgraph of $\Delta(G)$ on τ has no edge, $|\tau| \leq 2$ and $|\pi(M/N)| \geq 4$.

If $|\tau| \geq 3$, then there is an edge among two distinct primes in τ by Pálfy's Condition; and if $|\pi(M/N)| < 4$, then $|\pi(M/N)| = |\pi(G/N)| = 3$, which implies that $|\tau| = |\rho(G)| - |\pi(G/N)| = 3$. Thus we need to show that there is no edge among primes in τ. This can be proved using the same argument as in the proof of Claim 1 in Lemma 4.5.

Claim 2. $M/N \cong \mathrm{PSL}_2(q)$, where $q = p^f \geq 11$, p is a prime and $f \geq 1$.

If $|\pi(G/N)| \geq 5$, then the result follows from Lemma 4.2. Hence, we assume that $|\pi(G/N)| \leq 4$. Since $4 \leq |\pi(M/N)| \leq |\pi(G/N)| \leq 4$, we obtain that $|\pi(G/N)| = |\pi(M/N)| = 4$, and so $|\tau| = 2$. Writing $\tau = \{r, s\}$. As r and s are not joined to each other by Claim 1, $\{r, s\}$ is not an edge of any triangle in $\Delta(G)$ and thus the subgraph of $\Delta(G)$ on $\pi(G/N)$ has no triangle and so $\Delta(G/N)$ has no triangle. Now the result follows from [15, Lemma 3.2].

Claim 3. $\rho(G) = \pi(G/N)$.

It suffices to show that $\tau = \emptyset$. By way of contradiction, assume that τ is nonempty and let $r \in \tau$. By applying [15, Lemma 4.2], we deduce that $\{r, p_1, p_2\}$ is the vertex set of a triangle in $\Delta(G)$, where $\{p_1, p_2\} \subseteq \pi(M/N)$. Since r has degree three in $\Delta(G)$ and r is not adjacent to any prime in τ, it must be adjacent to some prime $p_3 \in \pi(G/N)$. Hence $rp_3 \mid \chi(1)$ for some $\chi \in \mathrm{Irr}(G)$. It follows that $\pi(\chi(1)) = \{r, p_3\}$ since $\{r, p_3\}$ is not an edge of any triangle in $\Delta(G)$. Let $\psi \in \mathrm{Irr}(M)$ be an irreducible constituent of χ_M and let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of ψ_N. Then θ is an irreducible constituent of χ_N. Since $r \nmid |G/N|$, we deduce that $r \mid \theta(1)$. As $|\pi(M/N)| \geq 4$, we deduce that θ is not extendible to M and thus $\psi(1)/\theta(1) = p_3^a$ for some integer $a \geq 1$.

Assume first that θ is not M-invariant. Then $I = I_M(\theta) \lneq M$. Let $\phi \in \mathrm{Irr}(I|\theta)$ such that $\phi^M = \psi$. We have that $\phi_N = e\theta$ for some integer $e \geq 1$. Now $\psi(1) = \phi^M(1) = |M : I|e\theta(1)$, so $|M : I|e = p_3^a$. In particular, $|M : I|$ is a power of p_3. Using [6, Hauptsatz II.8.27], we can deduce that either $q = 11$ and $I/N \cong A_5$ with $|M : I| = 11$ or $|M : I| = q + 1$ is a power of p_3 and I/N is the normalizer in M/N of a Sylow p-subgroup of $M/N \cong \mathrm{PSL}_2(q)$, where q is a power of a prime p. In both cases, we have that $\gcd(|M : I|, |I : N|) = 1$ and I/N is nonabelian. Thus if $e > 1$ and $u \mid e$ is a prime, then since $e \mid |I/N|$, $e \neq p_3$, which is impossible. Therefore, $e = 1$, which implies that ϕ is an extension of θ to I. Since I/N is nonabelian, we can find $\lambda \in \mathrm{Irr}(I/N)$ with $\lambda(1) > 1$. By Gallagher's Theorem and Clifford's Theory, we obtain that $(\phi\lambda)^M \in \mathrm{Irr}(M)$ and so $(\phi\lambda)^M(1) = |M : I|\lambda(1)\theta(1) \in \mathrm{cd}(M|\theta)$. Since $\lambda(1) > 1$ and $\lambda(1) \mid |I/N|$, we can find a prime divisor v of $\lambda(1)$ such that $v \neq p_3$ and hence $\{r, p_3, v\}$ is the vertex set of some triangle in $\Delta(G)$, which is impossible.

Therefore, we can assume that θ is M-invariant but not extendible to M. Since $q \geq 11$, the Schur representation group of M/N is $\mathrm{SL}_2(q)$ and thus by the theory of character triple isomorphisms in [8, Chapter 11], we deduce that $\theta(1)(q \pm 1) \in \mathrm{cd}(M|\theta)$. In particular, r is adjacent to every prime in $\pi(q^2 - 1)$. It follows that $|\pi(q^2 - 1)| = 3$ since $|\pi(M/N)| \geq 4$ and the degree of r in $\Delta(G)$ is three. However, as $q \geq 11$ is odd, we see that the subgraph of $\Delta(G)$ on $\{r\} \cup \pi(q^2 - 1)$ has two triangles sharing a common edge, which is impossible.

The final contradiction.

By the previous claim, we have that $|\pi(G/N)| = |\rho(G)| = 6$. By Lemma 4.2, $G = M$ and $G/N \cong \mathrm{PSL}_2(2^f)$ with $f \geq 10$. Writing $\pi(2^f + \delta) = \{p_k\}_{k=1}^3$ and $\pi(2^f - \delta) = \{p_k\}_{k=4}^5$, where $\delta = \pm 1$.

Since $\Delta(G)$ is connected but $\Delta(G/N)$ is not, we deduce that N is nontrivial. Observe that in the graph $\Delta(G)$, the prime 2 is adjacent to exactly one prime in $\{p_k\}_{k=1}^3$, say p_i. Hence, there exists $\chi \in \mathrm{Irr}(G)$ with $2p_i \mid \chi(1)$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N. Since $\{2\}$ is isolated in $\Delta(G/N)$, we deduce that $\theta \neq 1_N$. As $\Delta(G)$ has no triangle which contains an edge with vertex set $\{2, p_i\}$, we must have that $\pi(\chi(1)) = \{2, p_i\}$.

Assume first that θ is not G-invariant and put $I = I_G(\theta)$. Then I/N is a proper subgroup of the simple group $G/N \cong \mathrm{PSL}_2(2^f)$. It follows that $|G : I|$ is divisible by the index of some maximal subgroup of G/N. Hence, $|G : I|$ is divisible by one of the numbers in (1). Now the first three possibilities cannot occur since $\chi(1)$ and hence $|G : I|$ is divisible by only one odd prime. For the last case, if $f = 2b$, then the odd part of last index is $2^f + 1$ which is also divisible by two distinct odd primes; therefore, we can assume that $f/b = n \geq 3$. In this case, $(2^{2f} - 1)/(2^{2b} - 1)$ must be a power of p_i, which is impossible by applying [15, Lemma 2.4] since $f \geq 10$.

Assume now that θ is G-invariant. Since the Schur multiplier of $G/N \cong \mathrm{PSL}_2(2^f)$ with $f \geq 10$ is trivial, we deduce from [8, Theorem 11.7] that θ is extendible to $\theta_0 \in \mathrm{Irr}(G)$. By applying Gallagher's Theorem [8, Corollary 6.17], we have that

$$\chi(1) \in \{\theta_0(1) = \theta(1), 2^f \theta(1), (2^f + \delta)\theta(1), (2^f - \delta)\theta(1)\} \subseteq \mathrm{cd}(G).$$

As $\chi(1)$ is divisible by only one odd prime, we deduce that $\chi(1) = \theta(1)$ or $2^f \theta(1)$. In both cases, we deduce that $p_i \mid \theta(1)$ and thus $(2^f - \delta)\theta(1)$ is divisible by p_4, p_5 and p_i, and so these three primes form a triangle in $\Delta(G)$, which is impossible. The proof is now complete. $\qquad\square$

Proof of Theorem A. Let G be a group with prime graph $\Delta(G)$. If $\Delta(G)$ is a complete square, then it is a cubic graph. Conversely, if $\Delta(G)$ is a cubic graph, then it is a complete square by Theorems 3.2 and 4.6. $\qquad\square$

5. EXAMPLES

Let H and K be groups such that $\rho(H) \cap \rho(K) = \emptyset$ and both $\Delta(H)$ and $\Delta(K)$ are disconnected graphs of order two. Let $G = H \times K$. Then $\Delta(G)$ is a square and G is solvable. Conversely, every group whose prime graph is a square must be a direct product of two groups H and K satisfying the properties above. (See [11, Corollary C]).

Based on this example, for each even integer $k \geq 4$, we can construct a solvable group G whose prime graph is k-regular with $|\rho(G)| = k + 2$. Writing $k = 4\ell + r$ with $r \in \{0, 2\}$ and let $n = k + 2$. Let $G_i, i = 1, 2, \cdots, \ell$, be groups whose prime graphs are squares such that $\rho(G_i) \cap \rho(G_j) = \emptyset$, for all $1 \leq i \neq j \leq \ell$. Let G_0 be a group whose prime graph $\Delta(G_0)$ is a square if $r = 2$ and is a disconnected graph with two vertices if $r = 0$, where $\rho(G_0) \cap (\cup_{i=1}^\ell \rho(G_i)) = \emptyset$. It follows that $|\rho(G_0)| = r + 2$. Let $G = \prod_{i=0}^\ell G_i$ be the direct product of all $G_i's$, for $0 \leq i \leq \ell$.

Then $|\rho(G)| = r + 2 + 4\ell = k + 2$ and it is not hard to verify that $\Delta(G)$ is k-regular and since each $G_i, 0 \leq i \leq \ell$ is solvable, we deduce that G is solvable. Fig. 5 gives an example of a 4-regular graph of order six constructed in this way.
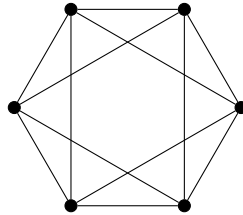

FIGURE 5. A quartic graph of order six

In view of Theorem A, Proposition 2.7 and the examples above, we formulate the following conjecture.

Conjecture. *Let G be a group and let $k \geq 2$ be an integer. Suppose that $\Delta(G)$ is k-regular. Then*

(1) *If $k \geq 5$ is odd, then $\Delta(G)$ is a complete graph of order $k + 1$.*
(2) *If $k \geq 4$ is even, then $\Delta(G)$ is either a complete graph of order $k + 1$ or a k-regular graph of order $k + 2$.*
(3) *If $|\rho(G)| = k + 2$, then G is solvable.*

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