Chapter 4 – Additional Topics with Functions

Content:

Section 4.2 Combining Functions; Composite Functions

When functions are combined using algebraic operations (+, -, * and /), the result is a function.

Example from section 1.3 is a profit (P) represented as a difference between revenue (R) and cost (C).
For R(x) = 20x and C(x) = 10x + 1000, profit P(x) is computed as:
P(x) = R(x) - C(x) = 20x - (10x + 1000) = 20x - 10x - 1000 = 10x - 1000

As opposed to combined functions where each function independently works on the same domain (x-value), **composite functions work on the output of the previous function**:
$(f \circ g)(x) = f(g(x))$
In this case, f works on $g(x)$, the output of g function.

As a result, computation is done from the inside out $g(x)$ must be computed first and then computed value is plugged into f.

Example: If f(x) = 3x and g(x) = x-1 , compute $(f \circ g)(5)$

$(f \circ g)(5) = f\big(g(5)\big)$
$g(5) = 5 - 1 = 4$
$f\big(g(5)\big) = f(4) = 3 * 4 = 12$, so the final answer is $f\big(g(5)\big) = 12$

Order of functions often makes the difference in composite functions: $(f \circ g)(x)$ is often not the same as $(g \circ h)(x)$

Example: If f(x) = 3x and g(x) = x - 1 , compute $(g \circ f)(5)$

$(g \circ f)(5) = g(f(5))$
$f(5) = 3 * 5 = 15$
$g(f(5)) = g(15) = 15 - 1 = 14$, so the final answer is $g(f(5)) = 14$

Example - the difference between combined and composite functions.
Just read it. There is no need to put this into your notes.
A person (x) needs to do hairdo (h) and manicure (m) – hairdo and manicure are two functions. When these two functions operate on the same person (x), the result is a manicured person with a nice hairdo (r). Adding two functions makes them combined functions.

$$r(x) = h(x) + m(x)$$

Now consider washing the hair (w) and blow-dry (b). These two functions done on the same person (x) represent composite functions and, depending on the order, they will produce two different results:

$$(b \circ w)(x) = nice\ hhhhhhh\ hhhhairdo$$
$$(w \circ b)(x) = wet\ hhhhair$$

Example 7. If $f(x) = \sqrt{x-5}$ and g(x) = $2x^2$ - 4.
c. Compute $(f \circ g)(-3)$ and $(g \circ f)(9)$ without using a calculator.

$(f \circ g)(-3) = f(g(-3))$
g(-3)= $2*(-3)^2$-4=2*9-4=18-4=14
$f(14) = \sqrt{14-5} = \sqrt{9}$ =3
The answer is: $(f \circ g)(-3) = 3$

$(g \circ f)(9) = g(f(9))$

$f(9) = \sqrt{9-5} = \sqrt{4} = 2$

g(2)=$2*2^2$-4=8-4=4

The answer is: $(g \circ f)(9) = 4$

Example: If $f(x) = \sqrt{x-5}$ and g(x) = 2x² - 4 compute $(g \circ f)(-3)$

$(g \circ f)(-3) = g(f(-3))$

$f(-3) = \sqrt{-3-5} = \sqrt{-8}$ this not defined in rational numbers.

Since $f(-3)$ is not defined, $g(f(-3))$ is also not defined.

We found out that $(f \circ g)(-3) = 3$ while $(g \circ f)(-3)$ is not defined.

This shows two things:

1. The order in which the functions are applied usually produces different results.
2. The domains of both f and g affect the result of composite functions.

This is why the definition of composite functions listed below must include a statement about the domains as well.

The **composite function** f **of** g is denoted by $f \circ g$ and is defined as:

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the subset of the domain for g for which $f \circ g$ is defined.

also

$$(g \circ f)(x) = g(f(x))$$

The domain of $g \circ f$ is the subset of the domain for f for which $g \circ f$ is defined.

ICE – 4.2:21 Use *f(x)*=2x² and $g(x) = \frac{x-5}{3}$ to evaluate composite functions.

Also, discuss the domains.

a. $(f \circ g)(2)$ b. $(g \circ f)(-2)$

So far we computed the values for composite functions but we can leave the variable as is, to get a generic composite function:

Example 6. Find $(h \circ f)(x)$ using $f(x) = 2x - 5$ and $h(x) = \frac{1}{x}$

(h ∘ f)(x) = *h(f(x)*

f(x *)=2x-5*

$h(2x\text{-}5) = \frac{1}{2x-5}$

The answer is: $(h \circ f)(x) = \frac{1}{2x-5}$

Section 4.3 Inverse *Functions*

Composite functions where order does not matter and the result is identity function (y=x) are inverse functions.

Functions f and g for which $f\left(g(x)\right) = x$ for all x in the domain of g

and

$g\left(f(x)\right) = x$ for all x in the domain of f

are called **inverse functions**.

If that is the case, then g can be denoted as f^{-1} (read f **inverse**).

Note that in this case, -1 does not represent the exponent. In other words: $f^{-1} \neq \frac{1}{f(x)}$

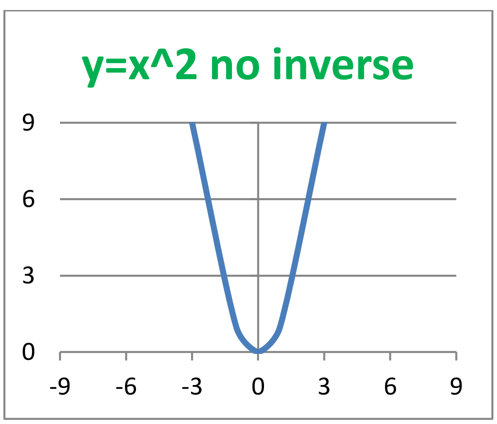
How do we know if a function f has an inverse function f^{-1}?

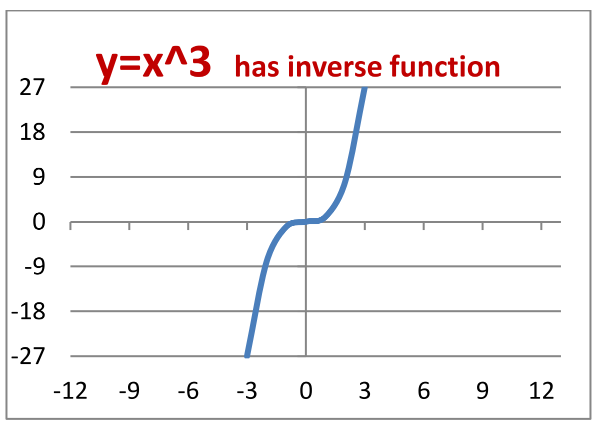
If f is a one-to-one function then it has a f^{-1}.

A **one-to-one function** has **for every element of the range only one element in the domain.**

Example of the function that does not have the inverse function is $f(x) = x^2$

Example of the function that has an inverse function is $f(x) = x^3$





If no horizontal line intersects the graph of a function in more than one point then the function is a one-to-one function (and therefore it has an inverse function).

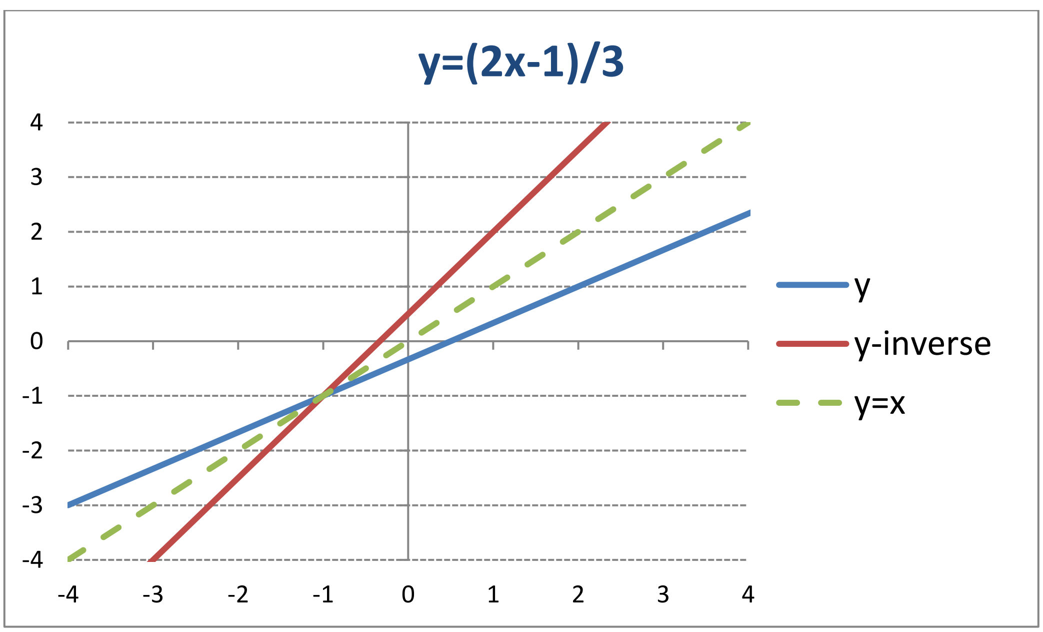
Steps to find the inverse function:

1. Rewrite the equation replacing f(x) with y
2. Interchange x and y in the equation.
3. Solve the new equation for y. If the solution is not unique then there is no inverse function.
4. Replace y with $f^{-1}(x)$

Example 3. a. Find the inverse function of $f(x) = \frac{2x-1}{3}$

a. Graph $f(x) = \frac{2x-1}{3}$ and its inverse function on the same axes.

Rewrite the equation replacing $f(x)$ with y	$y = \frac{2x-1}{3}$
Interchange x and y in the equation	$x = \frac{2y-1}{3}$
Solve the new equation for y	$3x = 2y - 1 \quad x + 1 = 2y \quad \frac{3x+1}{2} = y$
Replace y with $f^{-1}(x)$	$f^{-1}(x) = \frac{3x+1}{2}$

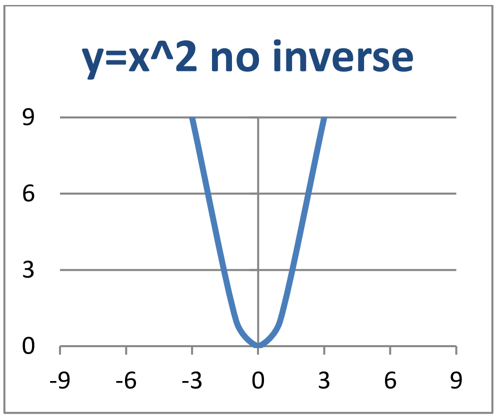


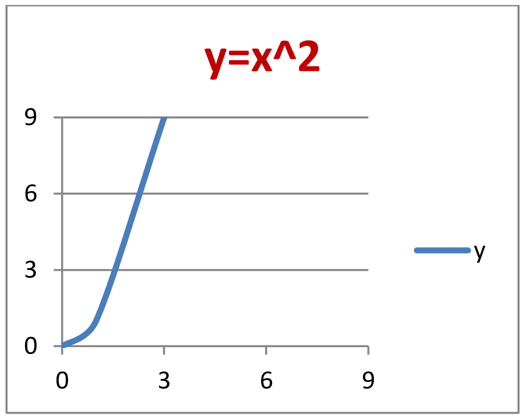
Graphs of inverse functions are symmetrical with respect to the line y=x.

Inverse functions on limited domains

Some functions may not have inverse functions when the domain (D) is all real numbers (R) but if the domain is a subset of R where they are on-to-one function, on that **limited domain** they have an inverse function.

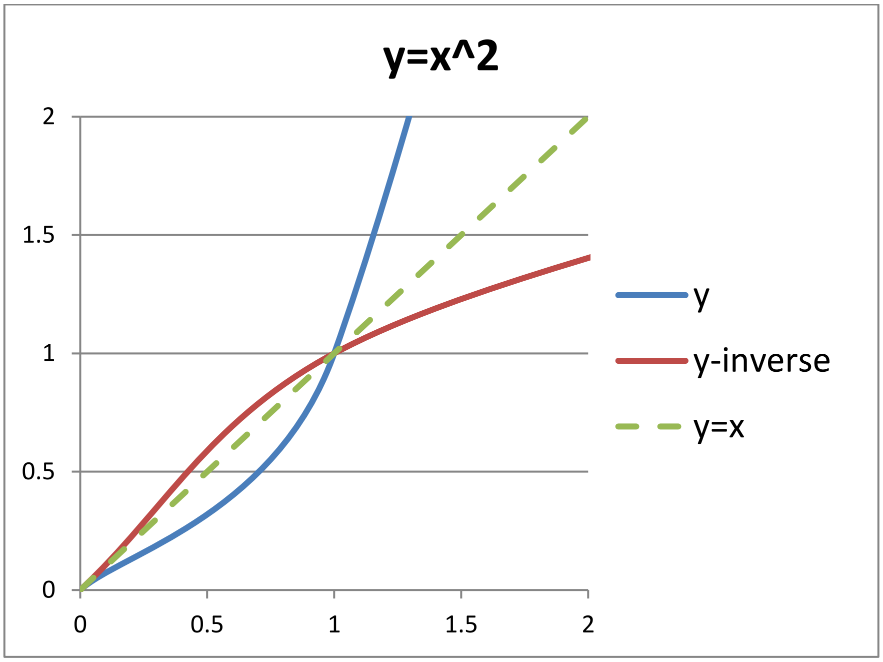
Example: The function $f(x) = x^2$ on $D = R$ (or $x \in R$) has no inverse *function.*
The function $f(x) = x^2$ on $\boldsymbol{D} = [\mathbf{0}, \boldsymbol{\infty})$ (or $\boldsymbol{x} > 0$) is one-to-one and has the inverse function.





So if the starting function is: $f(x) = x^2$ where a domain is restricted to $x > 0$ steps to find inverse function are the same as before only we must carry on domain restriction as well.

Rewrite the equation replacing $f(x)$ with y	$y = x^2$ where x>0
Interchange x and y in the equation and restriction.	$x = y^2$ where y>0
Solve the new equation for y	$y = \pm\sqrt{x}$ but since y>0 final solution is $y = \sqrt{x}$ and x>0 because of even root
Replace y with $f^{-1}(x)$	$f^{-1}(x) = \sqrt{x}$ x>0



ICE: 4.3:16 Find inverse function for $g(x) = 4x + 1$

$y = 4x + 1 \quad x = 4y + 1 \quad x + 1 = 4y \quad y = \frac{x+1}{4} \quad \boldsymbol{g}^{-1} = \frac{x+1}{4}$